No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$. You can use $\binom{k}{n}$ in your answers without having to explain what that symbol means.

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1 (13 pts). Find the orthogonal trajectories of the family of curves $xy = k$.

2 (24 pts). This question is about the curve given parametrically by $x = 3 - t^2$, $y = t^2 - t$.

a. Express $\dfrac{dy}{dx}$ on this curve as a function of t.

b. Find the equation of the line tangent to the curve at the point corresponding to $t = 1$.

c. Express $\dfrac{d^2y}{dx^2}$ on this curve as a function of t.

d. Find the length of the curve from $t = 0$ to $t = 2$. Express your answer as a definite integral, but **do not evaluate**.

3 (12 pts). Eliminate the parameter to find a Cartesian equation of the curve given by the parametric equations $x = \sqrt{1-t}$, $y = t + 1$ for $t \leq 1$. Sketch the curve and indicate with an arrow the direction it is traced as t increases.

4 (13 pts). Sketch the curve given by the polar equation $r = 2\sin 3\theta$. Label θs as necessary to clearly show where r attains a maximum or a minimum or is zero.

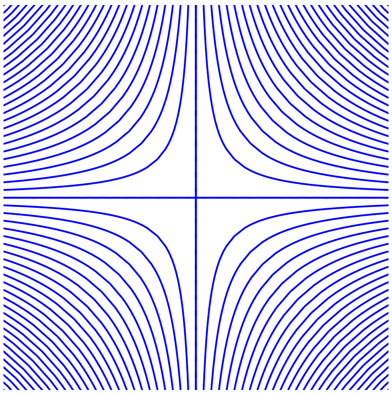
5 (20 pts). Find a power series expansion for the given function or integral. (Do not use a binomial series in part c.)

a. $x^2 \cos x$ b. $\sqrt[5]{(1+x)}$ c. $\displaystyle\int \frac{1}{1+2x^3}\,dx$

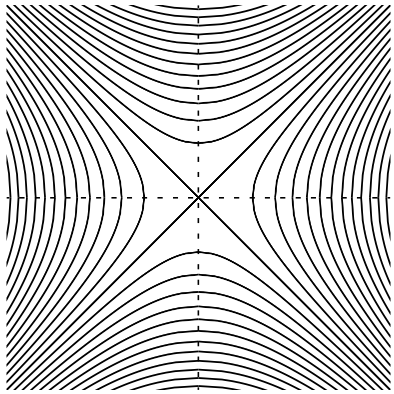
6a (9 pts). Find the third degree Taylor polynomial of e^{-x} centered at $a = 1$.

6b (9 pts). Find an upper bound for the absolute error occurring when e^{-x} is approximated on the interval $[0.8, 1.2]$ by its third degree Taylor polynomial centered at $a = 1$.

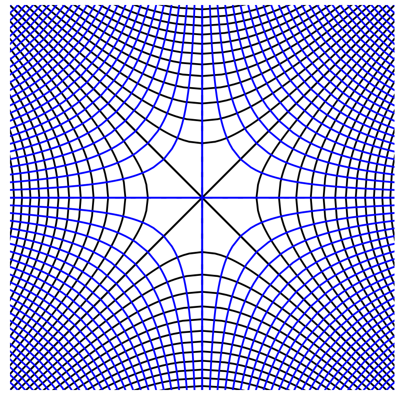
1. (Source: 9.3.31) First use implicit differentiation to find $\frac{dy}{dx}$ along the family $xy = k$. Differentiate both sides with respect to x: $y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$. So, along the orthogonal trajectories, $\frac{dy}{dx} = \frac{x}{y}$. Use separation of variables to solve this equation.
$y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$, or, since $2C$ is a constant, $y^2 = x^2 + k$.
(Here's a graph of the original family, its orthogonal trajectories, and two together.)



$xy = k$



$y^2 = x^2 + k$



Both

2. (Source: 10.2.11, 41) a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-1}{-2t}$, or $-1 + \frac{1}{2}t^{-1}$

b. At $t = 1$, the slope is $\frac{dy}{dx} = -1 + \frac{1}{2}1^{-1} = -\frac{1}{2}$. The point of tangency is $(x(1), y(1)) = (2, 0)$, so the point-slope form of the line is $y = -\frac{1}{2}(x - 2)$.

c. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{\frac{d}{dt}\frac{dy}{dx}}{dx/dt} = \dfrac{-\frac{1}{2}t^{-2}}{-2t}$, or $\frac{1}{4}t^{-3}$.

d. $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(-2t)^2 + (2t-1)^2}\,dt$, so the length of the curve is $\int_0^2 \sqrt{(-2t)^2 + (2t-1)^2}\,dt$, or $\int_0^2 \sqrt{8t^2 - 4t + 1}\,dt$.

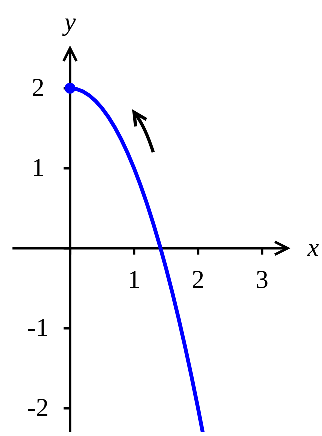
3. (Source: 10.1.9) Solve for t in either equation and substitute it into the other. Either

$$t = y - 1 \Longrightarrow x = \sqrt{1 - (y-1)} = \sqrt{2 - y}$$

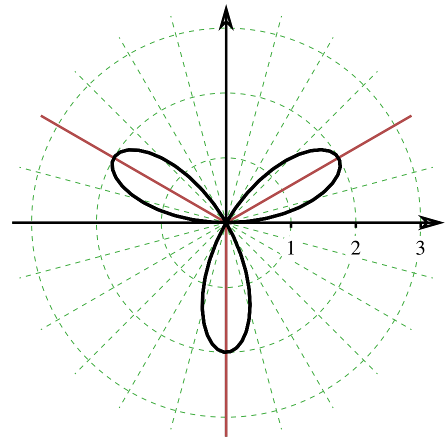
or

$$t = 1 - x^2 \Longrightarrow y = (1-^2) + 1 = 2 - x^2 \quad (x \geq 0).$$

In the second parameter-free equation, note that $x \geq 0$ because x is a square root. The graph is the right half of a parabola. As t increases, x decreases and y increases, so the curve is traced up and to the left, ending at $(0, 2)$ at $t = 1$:



4. (Source: 10.3.37) The graph is a 3-leafed rose. $r = 2$ at $\theta = \pi/6$, $5\pi/6$, and $9\pi/6 = 3\pi/2$, and $r = -2$ at $\theta = 3\pi/6$, $7\pi/6$, and $11\pi/6$. The curve reaches the origin when $r = 0$,and this is at the integer multiples of $\pi/3$.



5a. (Source: 11.10.33) Use the MacLauren series expansion for $\cos x$:
$x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!}$.

5b. (Source: 11.9.25, 28) Binomial series: $\sqrt[5]{(1+x)} = (1+x)^{1/5} = \sum_{n=0}^{\infty} \binom{1/5}{n} x^n$.

5c. (Source: 11.9.23) Find a series for the integrand by using the geometric series:
$\frac{1}{1+2x^3} = \frac{1}{1-(-2x^3)} = \sum_{n=0}^{\infty}(-2x^3)^n = \sum_{n=0}^{\infty}(-2)^n x^{3n}$. Now integrate:
$\int \frac{1}{1+2x^3}\,dx = \int \left(\sum_{n=0}^{\infty}(-2)^n x^{3n}\right)\,dx = \sum_{n=0}^{\infty}(-2)^n(\int x^{3n}\,dx) = \sum_{n=0}^{\infty} \frac{(-2)^n x^{3n+1}}{3n+1} + C$.

6a. (Source: 11.11.15) Here's a table of the first 4 coefficients of the Taylor series.

n	$f^{(n)}(x)$	$f^{(n)}(-1)/n!$
0	e^{-x}	e^{-1}
1	$-e^{-x}$	$-e^{-1}$
2	e^{-x}	$e^{-1}/2$
3	$-e^{-x}$	$-e^{-1}/6$

The third degree Taylor polynomial is

$$T_3(x) = e^{-1} - e^{-1}(x-1) + \frac{1}{2}e^{-1}(x-1)^2 - \frac{1}{6}e^{-1}(x-1)^3.$$

6b. (Source: 11.11.15) On the interval $[0.8, 1.2]$, $|f^{(4)}(x)| = e^{-x} \le e^{-0.8}$, so according to Taylor's Theorem,

$$|e^{-x} - T_3(x)| = \frac{|f^{(4)}(c)||x-1|^4}{4!} \le \frac{e^{-0.8}}{4!}(0.2)^4$$