1 (13 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. 
\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^4 - 1}} \]

2 (8 pts). Find the sum of the series, if it exists. 
\[ \sum_{n=1}^{\infty} \frac{3^{n-1}}{n^{n+1}} \]

3 (12 pts). Determine whether the series converges or diverges. 
\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]

4 (20 pts). Determine whether the series converges absolutely, converges conditionally, or diverges.

a. \[ \sum_{n=0}^{\infty} \frac{n!}{3^n} \]

b. \[ \sum_{n=0}^{\infty} \left(\frac{1-2n}{3n+1}\right)^n \]

5 (10 pts). \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] converges. If \( s \) is the sum of this series and \( s_{20} \) is its 20th partial sum, bound \( s - s_{20} \) above and below.

6 (10 pts). \[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} \] converges. If \( \sigma \) is the sum of this series and \( \sigma_n \) is its \( n \)th partial sum, how large must \( n \) be to ensure that \( |\sigma - \sigma_n| \leq 10^{-6} \)?

7 (13 pts). Find the **radius** of convergence of the power series: 
\[ \sum_{n=1}^{\infty} \frac{5^n(x-3)^n}{n^2} \]
(You are not required to find the **interval** of convergence.)

8 (14 pts). The radius of convergence of the power series \( \sum_{n=1}^{\infty} e^{1/n} (x+2)^{2n} \) is 1. Find its interval of convergence, including which of the endpoints (if any) are included in the interval.
1. (Source: 11.6.10) Test for absolute convergence first. \( \sum_{n=2}^{\infty} \frac{(−1)^n n}{\sqrt[n]{4}} = \sum_{n=2}^{\infty} \frac{n}{\sqrt[n]{4}}. \) Since the integral diverges, so does the series.

To test for conditional convergence, use the Alternating Series Test. Let \( b_n = \frac{n}{\sqrt[n+1]{4}}. \)

Then \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n}{n^2} = \lim_{n \to \infty} \frac{1}{n} = 0. \)

To see that \( b_n \) decreases, take its derivative with respect to \( n \):

\[
b'_n = \frac{(n^4 - 1)^{1/2} - n^4 4n^3(n^4 - 1)^{-1/2}}{(n^4 - 1)} = \frac{n^4 - 1 - 2n^4}{(n^4 - 1)^{3/2}} = \frac{-1 - n^4}{(n^4 - 1)^{3/2}}
\]

which is negative as long as it is defined, so certainly for \( n \geq 2 \) as in this sum. Since \( b_n \) is decreasing and goes to zero, \( \sum_{n=2}^{\infty} \frac{(−1)^n n}{\sqrt[n+1]{4}} \) converges, hence is conditionally convergent.

(You could also show that \( b_n \) is decreasing without a derivative, using

\[
b_n = \frac{1}{n \sqrt{1 - \frac{1}{n^2}}}
\]

from above. As \( n \) increases, \( \frac{1}{n^2} \) decreases, so \( 1 - \frac{1}{n^2} \) increases, and therefore \( n \sqrt{1 - \frac{1}{n^2}} \) also increases, so \( \frac{1}{n \sqrt{1 - \frac{1}{n^2}}} \) decreases.)

2. (Source: 11.2.19) \( \sum_{n=1}^{\infty} \frac{3^n}{\pi^n} = \sum_{n=1}^{\infty} \frac{1}{\pi^n} \cdot \frac{3^n}{\pi^n} = \sum_{n=1}^{\infty} \frac{1}{\pi^n} \cdot (\frac{3}{\pi})^n. \) This series is geometric with \( r = 3/\pi, \) and its first term is \( 1/\pi^2. \) Since \( |r| < 1, \) the sum of the series is \( \frac{\frac{1}{\pi^2}}{1-\frac{3}{\pi^2}} = \frac{\pi^2}{\pi^2 - 3}. \)

3. (Source: 11.3.21) Since \( f(x) = \frac{1}{\ln x} \) is positive and decreasing, the Integral Test says that the series \( \sum_{n=2}^{\infty} \frac{1}{\ln n} \) and the improper integral \( \int_{2}^{\infty} \frac{dx}{x \ln x} \) either both converge or both diverge.

In the integral, substitute \( u = \ln x, \) \( du = \frac{1}{x} \) \( dx. \) As \( x \to \infty, \) so does \( u, \) and so the integral equals

\[
\int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{B \to \infty} \int_{\ln 2}^{B} \frac{du}{u} = \lim_{B \to \infty} (\ln u)_{\ln 2}^{B} = \lim_{B \to \infty} (\ln B - \ln(\ln 2)) = \infty.
\]

Since the integral diverges, so does the series.

4a. (Source: 11.6.8) Use the ratio test.

\[
\lim_{n \to \infty} \frac{(n+1)!}{3n+1} = \lim_{n \to \infty} \frac{(n+1)!}{n!} \cdot \frac{3^n}{3n+1} = \lim_{n \to \infty} \frac{n+1}{3n+1} = \infty > 1,
\]

so the series diverges.

4b. (Source: 11.6.21, 22) Use the root test:

\[
\lim_{n \to \infty} \left( \frac{1 - 2n}{3n + 1} \right)^{1/n} = \lim_{n \to \infty} \left| \frac{1 - 2n}{3n + 1} \right| = \lim_{n \to \infty} \frac{2n - 1}{3n + 1} = \frac{2}{3} < 1,
\]

so the series is absolutely convergent.
5. (Source: 11.3.more.1b) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{20} \frac{1}{n^2} = \sum_{n=21}^{\infty} \frac{1}{n^2}, \] and by the integral test, \[ \int_{21}^{\infty} \frac{dx}{x^2} \leq \sum_{n=21}^{\infty} \frac{1}{n^2} \leq \int_{20}^{\infty} \frac{dx}{x^2}. \]

The first integral is \[ \lim_{C \to \infty} \int_{21}^{C} x^{-2} \, dx = \lim_{C \to \infty} -\frac{1}{C} \bigg|_{21}^{C} = \lim_{C \to \infty} \left( -\frac{1}{C} + \frac{1}{21} \right) = \frac{1}{21}. \]
Likewise, the second integral is \[ \frac{1}{20}, \] so \[ \frac{1}{21} \leq s - s_{20} \leq \frac{1}{20}. \]

6. (Source: 11.5.more.1b) By the alternating series test, \[ \sigma \text{ is between any two consecutive partial sums } \sigma_n \text{ and } \alpha_{n+1}, \] so

\[ |\sigma - \sigma_n| \leq |\sigma_{n+1} - \sigma_n|. \]

The difference between these is the \((n+1)\)th term of the series:

\[ |\sigma - \sigma_n| \leq |\sigma_{n+1} - \sigma_n| = \left| \frac{(-1)^n}{(n+1)^2} \right| = \frac{1}{(n+1)^2}, \]

so, to make \( |\sigma - \sigma_n| \leq 10^{-6} \), just make \( \frac{1}{(n+1)^2} \leq 10^{-6} \), or

\[ 10^6 \leq (n+1)^2 \Rightarrow 10^3 \leq n+1 \Rightarrow 999 \leq n. \]

7. (Source: 11.8.9) Root or ratio will work. Here’s root:

\[ \lim_{n \to \infty} \left| \frac{(-1)^n 5^n (x-3)^n}{n^2} \right|^{1/n} = \lim_{n \to \infty} \left( \frac{5^n |x-3|^n}{n^2} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{5|x-3|}{n^{2/n}} \right). \] (1)

By the Famous Limit Everyone Should Know \( \lim_{n \to \infty} n^{p/n} = 1 \) for all \( p \), so (1) equals \( 5|x-3| \).

The series converges when \( 5|x-3| < 1 \), or \( |x-3| < 1/5 \). That’s all \( x \) less than 1/5 to the right or left of 3, so the radius of convergence is 1/5.

8. (Source: 11.2.27) Since the radius of convergence is 1 and the center of the series is \(-2\), the endpoints of the interval of convergence at \(-2 \pm 1 = -3 \) and \(-1 \). Test the series for convergence at both of these.

At \( x = -3 \), the series is \( \sum_{n=1}^{\infty} e^{1/n}(-3+2)^{2n} = \sum_{n=1}^{\infty} e^{1/n}(-1)^{2n} = \sum_{n=1}^{\infty} e^{1/n} \). Since \( \lim_{n \to \infty} e^{1/n} = e^0 = 1 \neq 0 \), this series diverges by the \( n \)th term test.

At \( x = -1 \), the series is \( \sum_{n=1}^{\infty} e^{1/n}(-1+2)^{2n} = \sum_{n=1}^{\infty} e^{1/n} \) again, so the series diverges at \( x = -1 \), and the interval of convergence is \((-3, -1)\).