No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

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1 (13 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\sqrt{n^4-1}}$

2 (8 pts). Find the sum of the series, if it exists. $\sum_{n=1}^{\infty} \frac{3^{n-1}}{\pi^{n+1}}$

3 (12 pts). Determine whether the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

4 (20 pts). Determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

b. $\sum_{n=0}^{\infty} \left(\frac{1-2n}{3n+1}\right)^n$

5 (10 pts). $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. If s is the sum of this series and s_{20} is its 20th partial sum, bound $s - s_{20}$ above and below.

6 (10 pts). $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges. If σ is the sum of this series and σ_n is its nth partial sum, how large must n be to ensure that $|\sigma - \sigma_n| \leq 10^{-6}$?

7 (13 pts). Find the **radius** of convergence of the power series: $\sum_{n=1}^{\infty} (-1)^n \frac{5^n (x-3)^n}{n^2}$
(You are not required to find the **interval** of convergence.)

8 (14 pts). The radius of convergence of the power series $\sum_{n=1}^{\infty} e^{1/n}(x+2)^{2n}$ is 1. Find its interval of convergence, including which of the endpoints (if any) are included in the interval.

1. (Source: 11.6.10) Test for absolute convergence first. $\sum_{n=2}^{\infty}\left|\frac{(-1)^n n}{\sqrt{n^4-1}}\right| = \sum_{n=2}^{\infty}\frac{n}{\sqrt{n^4-1}}$. Compare this series to $\sum_{n=2}^{\infty}\frac{n}{\sqrt{n^4}} = \sum_{n=2}^{\infty}\frac{1}{n}$. Since $0 \le \frac{1}{n} = \frac{n}{\sqrt{n^4}} \le \frac{n}{\sqrt{n^4-1}}$, and $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty}\frac{n}{\sqrt{n^4-1}}$, and $\sum_{n=2}^{\infty}\frac{(-1)^n n}{\sqrt{n^4-1}}$ fails to be absolutely convergent.

To test for conditional convergence, use the Alternating Series Test. Let $b_n = \frac{n}{\sqrt{n^4-1}}$. Then $\lim_{n\to\infty} b_n =$

$$\lim_{n\to\infty}\frac{n}{n^2\sqrt{1-\frac{1}{n^4}}} = \lim_{n\to\infty}\frac{1}{n\sqrt{1-\frac{1}{n^4}}} = 0.$$

To see that b_n decreases, take its derivative with respect to n:

$$b_n' = \frac{(n^4-1)^{1/2} - n\frac{1}{2}4n^3(n^4-1)^{-1/2}}{(n^4-1)}\cdot\frac{(n^4-1)^{1/2}}{(n^4-1)^{1/2}} = \frac{n^4-1-2n^4}{(n^4-1)^{3/2}} = \frac{-1-n^4}{(n^4-1)^{3/2}}$$

which is negative as long as it is defined, so certainly for $n \ge 2$ as in this sum. Since b_n is decreasing and goes to zero, $\sum_{n=2}^{\infty}\frac{(-1)^n n}{\sqrt{n^4-1}}$ converges, hence is conditionally convergent.

(You could also show that b_n is decreasing without a derivative, using

$$b_n = \frac{1}{n\sqrt{1-\frac{1}{n^4}}}$$

from above. As n increases, $\frac{1}{n^4}$ decreases, so $1-\frac{1}{n^4}$ increases, and therefore $n\sqrt{1-\frac{1}{n^4}}$ also increases, so $\frac{1}{n\sqrt{1-\frac{1}{n^4}}}$ decreases.)

2. (Source: 11.2.19) $\sum_{n=1}^{\infty}\frac{3^{n-1}}{\pi^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{3\pi}\cdot\frac{3^n}{\pi^n} = \sum_{n=1}^{\infty}\frac{1}{3\pi}\cdot\left(\frac{3}{\pi}\right)^n$. This series is geometric with $r = 3/\pi$, and its first term is $1/\pi^2$. Since $|r| < 1$, the sum of the series is $\frac{a}{1-r} = \frac{\pi^{-2}}{1-3\pi^{-1}} = \frac{1}{\pi^2-3\pi}$.

3. (Source: 11.3.21) Since $f(x) = \frac{1}{x\ln x}$ is positive and decreasing, the Integral Test says that the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ and the improper integral $\int_2^{\infty}\frac{dx}{x\ln x}$ either both converge or both diverge.

In the integral, substitute $u = \ln x$, $du = \frac{1}{x}\,dx$. As $x \to \infty$, so does u, and so the integral equals

$$\int_{\ln 2}^{\infty} = \frac{du}{u} = \lim_{B\to\infty}\int_{\ln 2}^{B} = \frac{du}{u} = \lim_{B\to\infty}(\ln u)\Big|_{\ln 2}^{B} = \lim_{B\to\infty}(\ln B - \ln(\ln 2)) = \infty.$$

Since the integral diveges, so does the series.

4a. (Source: 11.6.8) Use the ratio test.

$$\lim_{n\to\infty}\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{3^n}{3^{n+1}} = \lim_{n\to\infty}\frac{n+1}{3} = \infty > 1,$$

so the series diverges.

4b. (Source: 11.6.21,22) Use the root test:

$$\lim_{n\to\infty}\left(\left|\frac{1-2n}{3n+1}\right|^n\right)^{1/n} = \lim_{n\to\infty}\left|\frac{1-2n}{3n+1}\right| = \lim_{n\to\infty}\frac{2n-1}{3n+1} = \frac{2}{3} < 1,$$

so the series is absolutely convergent.

5. (Source: 11.3.more.1b) $\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{20} \frac{1}{n^2} = \sum_{n=21}^{\infty} \frac{1}{n^2}$, and by the integral test, $\int_{21}^{\infty} \frac{dx}{x^2} \le \sum_{n=21}^{\infty} \frac{1}{n^2} \le \int_{20}^{\infty} \frac{dx}{x^2}$.

The first integral is $\lim_{C\to\infty} \int_{21}^{C} x^{-2}\,dx = \lim_{C\to\infty} -x^{-1}\Big|_{21}^{C} = \lim_{C\to\infty}(-\frac{1}{C} + \frac{1}{21}) = \frac{1}{21}$. Likewise, the second integral is $\frac{1}{20}$, so $\frac{1}{21} \le s - s_{20} \le \frac{1}{20}$.

6. (Source: 11.5.more.1b) By the alternating series test, σ is between any two consecutive partial sums σ_n and α_{n+1}, so

$$|\sigma - \sigma_n| \le |\sigma_{n+1} - \sigma_n|.$$

The difference between these is the $(n+1)$th term of the series:

$$|\sigma - \sigma_n| \le |\sigma_{n+1} - \sigma_n| = \left|\frac{(-1)^n}{(n+1)^2}\right| = \frac{1}{(n+1)^2},$$

so, to make $|\sigma - \sigma_n| \le 10^{-6}$, just make $\frac{1}{(n+1)^2} \le 10^{-6}$, or

$$10^6 \le (n+1)^2 \Rightarrow 10^3 \le n+1 \Rightarrow 999 \le n.$$

7. (Source: 11.8.9) Root or ratio will work. Here's root:

$$\lim_{n\to\infty} \left|\frac{(-1)^n 5^n (x-3)^n}{n^2}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{5^n |x-3|^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{5|x-3|}{n^{2/n}}\right). \qquad (1)$$

By the Famous Limit Everyone Should Know $\lim_{n\to\infty} n^{p/n} = 1$ for all p, so (1) equals $5|x-3|$.

The series converges when $5|x-3| < 1$, or $|x-3| < 1/5$. That's all x less than $1/5$ to the right or left of 3, so the radius of convergence is $1/5$.

8. (Source: 11.2.27) Since the radius of convergence is 1 and the center of the series is -2, the endpoints of the interval of convegence at $-2 \pm 1 = -3$ and -1. Test the series for convergence at both of these.

At $x = -3$, the series is $\sum_{n=1}^{\infty} e^{1/n}(-3+2)^{2n} = \sum_{n=1}^{\infty} e^{1/n}(-1)^{2n} = \sum_{n=1}^{\infty} e^{1/n}$. Since $\lim_{n\to\infty} e^{1/n} = e^0 = 1 \ne 0$, this series diverges by the nth term test.

At $x = -1$, the series is $\sum_{n=1}^{\infty} e^{1/n}(-1+2)^{2n} = \sum_{n=1}^{\infty} e^{1/n}$ again, so the series diverges at $x = -1$, and the interval of convergence is $(-3, -1)$.