No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

---

1. This problem concerns approximations to the definite integral $\int_1^2 e^{1/x}\,dx$.
Leave unsimplified, unfinished arithmetic in your answers to parts a-c, but otherwise be completely explicit about how they are to be calculated. Your answer should not include "..." or anything else for me to fill in. On some parts, it might be helpful to know

$$|E_T| \leq \frac{K(b-a)^?}{12n^?} \qquad \text{and} \qquad |E_S| \leq \frac{K(b-a)^?}{180n^?}$$

1a (10 pts). Approximate the integral using Simpson's Rule with n = 6 subintervals. Repeat with the Trapezoid Rule. Label your answers so I can tell which is which.

1b (5 pts). I did some calculations and found that, if $f(x) = e^{1/x}$ and $1 \leq x \leq 2$, then

$$|f'(x)| \leq 5, \quad |f''(x)| \leq 10, \quad |f^{(3)}(x)| \leq 40, \quad |f^{(4)}(x)| \leq 300, \quad |f^{(5)}(x)| \leq 2000.$$

Assuming these are correct, how large might be the absolute error in the **Simpson's Rule** approximation in 1a?

1c (8 pts). If I require that the **Trapezoid Rule** approximate this definite integral with an absolute error less or equal 10^{-6}, how large must n be to guarantee this?

2a (17 pts). Find the area of the surface obtained by rotating the curve $y = \sqrt{1+2x}$, $0 \leq x \leq 1$ about the x axis.

2b (3 pts). Find the length of the curve in 2a. Express your answer as a definite integral, but **do not evaluate**.

3 (8 pts). Find the limit of the sequence, if it converges.

a. $e^{1/n} - (0.3)^n$ b. $\sqrt{\dfrac{n^2}{4n^2+1}}$

4 (8 pts). Find the average value of the function $r(x) = \dfrac{x}{x^2+2}$ on the interval $[1, 3]$.

5 (7 pts). A chain 20 ft long and weighing 30 lb is lying on the ground. How much work is required to raise one end of the chain up 8 ft? Express your answer as a definite integral, but **do not evaluate**.

6 (16 pts). Evaluate the improper integral, if it converges. $\int_0^\infty xe^{-2x}\,dx$

7 (18 pts). Let R be the area bounded by the two curves $y = 12 - x^2$ and $y = x^2 - 6$. Express the following as definite integrals, but **do not integrate**.

7a. The area of R.

7b. The volume of the solid obtained by rotating R about the line $y = -10$.

7c. The volume of the solid obtained by rotating R about the line $x = 10$.

7d. The volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares with one side in R.

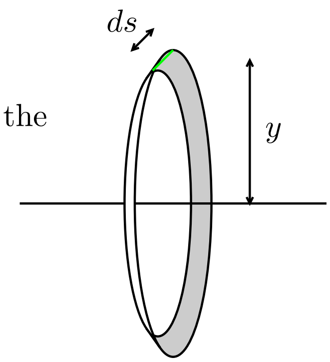
1a. (Source: 7.7.20) $\Delta x = (2-1)/6 = 1/6$ so subintervals have endpoints $1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}, \frac{11}{6}, 2$.

$$\texttt{TRAP} = \frac{1}{12}(e + 2e^{(6/7)} + 2e^{(6/8)} + 2e^{(6/9)} + 2e^{(6/10)} + 2e^{(6/11)} + e^{(1/2)}).$$
$$\texttt{SIMP} = \frac{1}{18}(e + 4e^{(6/7)} + 2e^{(6/8)} + 4e^{(6/9)} + 2e^{(6/10)} + 4e^{(6/11)} + e^{(1/2)}).$$

1b. In Simpson's Rule, K is an upper bound for $|f^{(4)}|$, so $|E_S| \le \frac{300(b-a)^5}{180n^4} = \frac{300\cdot 1^5}{180\cdot 6^4}$ (or $5 \cdot 3^{-1}6^{-4}$).

1c. In the Trapezoid Rule, K is an upper bound for $|f''(x)|$, so use $K = 10$. To ensure that $|E_T| \le 10^{-6}$, choose n so that $\frac{10\cdot(b-a)^3}{12n^2} \le 10^{-6}$:

$$\frac{10}{12n^2} \le \frac{1}{10^6} \Rightarrow \frac{5}{6}10^6 \le n^2 \Rightarrow 10^3\sqrt{\frac{5}{6}} \le n.$$



2a. (Source: 8.2.7) Let dA be the area of the ribbon swept out by the little piece of the curve at (x, y) as it it rotated about the x-axis.

$$dA = 2\pi y\, ds = 2\pi y\sqrt{1 + (\tfrac{dy}{dx})^2}\, dx$$
$$= 2\pi\sqrt{1+2x}\sqrt{1 + \left((1+2x)^{-1/2}\right)^2}\, dx = 2\pi\sqrt{1+2x+1}\, dx,$$

so the area is

$$\int_0^1 2\pi\sqrt{2x+2}\, dx = \tfrac{2\pi}{3}(2x+2)^{3/2}\Big|_0^1 = \tfrac{2\pi}{3}(4^{3/2} - 2^{3/2}) = \tfrac{2\pi}{3}(8 - 2^{3/2}).$$

2b. (Source: 8.1.3) $s = \int ds = \int_0^1 \sqrt{1 + \left((1+2x)^{-1/2}\right)^2}\, dx = \int_0^1 \sqrt{\frac{2x+2}{2x+1}}\, dx$.

3a. (Source: 11.1.17,21) $\lim_{n\to\infty}(e^{1/n} - (0.3)^n) = e^0 - 0 = 1$.

3b. (Source: 11.1.24) $\lim_{n\to\infty} \frac{n^2}{4n^2+1} = \lim_{n\to\infty} \frac{1}{4+n^{-2}} = \frac{1}{4}$, and, because the square root is a continuous function, $\lim_{n\to\infty} \sqrt{\frac{n^2}{4n^2+1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

4. (Source: 6.5.12) $r_{\text{ave}} = \frac{1}{3-1}\int_1^3 \frac{x}{x^2+2}\, dx = \frac{1}{2}\frac{1}{2}\ln(x^2+2)\Big|_1^3 = \frac{1}{4}(\ln 11 - \ln 3)$.

5. (Source: 6.4.14) Note that the chain weights 3/2 lb per ft. Grab one end and start lifting. Let dw be the work to raise your end of the chain another dy ft when you've already lifted y ft of chain off the ground.

$$dw = (\text{weight of } y \text{ ft of chain}) \cdot dy = \tfrac{3}{2}y\, dy, \text{ so total work} = \int_0^8 \tfrac{3}{2}y\, dy.$$

6. (Source: 7.8.19) $\int_0^\infty xe^{-2x}\, dx = \lim_{B\to\infty}\int_0^B xe^{-2x}\, dx$. Integration by parts:

$$u = x \qquad dv = e^{-2x}\, dx \qquad \Rightarrow \qquad du = dx \qquad v = -\tfrac{1}{2}e^{-2x}$$

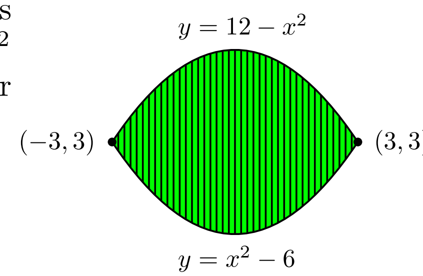
the definite integral becomes

$$\int_0^B xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_0^B + \int_0^B \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}Be^{-2B} - \frac{1}{4}e^{-2x}\Big|_0^B$$
$$= -\frac{1}{2}Be^{-2B} - \frac{1}{4}e^{-2B} + \frac{1}{4}.$$

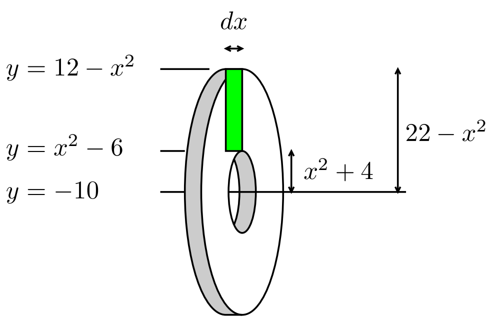
The improper integral is the limit of this as $B \to \infty$. The first of these terms is of the form "$\infty \cdot 0$" and requires l'Hospital's Rule. Rewriting that product as a quotient, $\lim_{B\to\infty} Be^{-2B} = \lim_{B\to\infty} \frac{B}{e^{2B}}$ which has the form "∞/∞." Apply l'Hospital's Rule to obtain $\lim_{B\to\infty} \frac{1}{2e^{2B}} = 0$. Therefore $\lim_{B\to\infty} Be^{-2B}$ also is zero, and

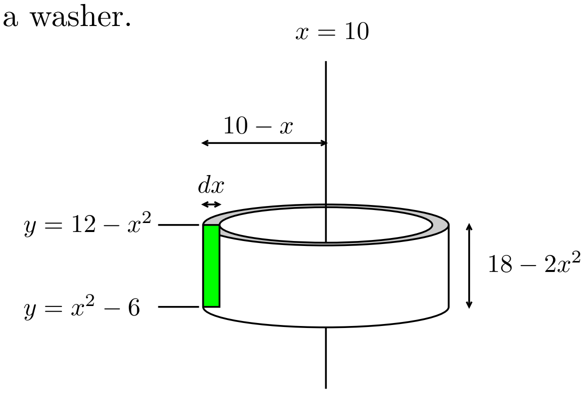
$$\int_0^\infty xe^{-2x}\,dx = \lim_{B\to\infty}\left(-\frac{1}{2}Be^{-2B} - \frac{1}{4}e^{-2B} + \frac{1}{4}\right) = \frac{1}{4}.$$

7 (1 pts). (Source: 6.1.13, 6.2.19-30, 6.3.17, 6.2.more.1h)
You don't need a highly accurate graph of R to answer this question. $y = x^2 - 6$ is a parabola that opens up and $y = 12 - x^2$ opens down. R must look something like this, after solving for intersection points and slicing vertically into rectangles.



a. The area of any one these rectangles is $((12 - x^2) - (x^2 - 6))\,dx$, so the area of R is $\int_{-3}^{3}(18 - 2x^2)\,dx$.
When we rotate each rectangle about the horizontal line $y = -10$, the result is a washer. Rotating about the vertical line $x = 10$ gives a washer.





b. $V = \int_{-3}^{3} \pi\big((22 - x^2)^2 - (x^2 + 4)^2\big)\,dx$

c. $V = \int_{-3}^{3} 2\pi(10 - x)(18 - 2x^2)\,dx$

d. In this non-rotational solid, each slice perpendicular to the x-axis has a square face and an infinitesimally small thickness. Let dV be the volume of the slice at position x. Then

$$V = \int dV = \int_{-3}^{3}(18 - 2x^2)^2\,dx$$