MATH 220–01 (Kunkle), Exam 1 Name: ______________________

100 pts, 75 minutes Feb. 2, 2016

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. **(Generally, this means to combine like terms, simplify exponents, add fractions that already have the same denominator, cancel common factors in a fraction, etc. Raise your hand if you're not sure whether to perform some simplification.)**

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

A mistake early in your solution does not rule out your receiving full credit for later steps.

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1 (6 pts). Differentiate. $f(x) = \cosh(x^2)\sinh(x^3)$

2 (11 pts). Evaluate the definite integral. $\int_0^2 t\sinh t\,dt$

3 (11 pts). Evaluate the limit.

a. $\lim_{x\to 1} \frac{x^{2/3}-1}{x^{5/3}-1}$

b. $\lim_{x\to 0} \frac{e^x-e^{-x}-3x}{x-\sin x}$

4 (9 pts). Evaluate the indefinite integral. $\int \tan^4 x\sec^4 x\,dx$

5 (12 pts). Evaluate the indefinite integral. $\int \sin(4x)\cos(3x)\,dx$

6 (14 pts). Rewrite $\sin^2 x\cos^4 x$ as a sum of sinusoidal functions, but **do not integrate**.

7 (11 pts). Evaluate the indefinite integral. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

8 (6 pts). What trigonometric substitution would you make in the following integrals? Write your answer on the line, using the space below for any necessary work.
Do not carry out the substitution. Do not integrate.

a. $\int (x^2-3)^{7/2}\,dx$ $x =$ ____________ $dx =$ ____________

b. $\int (x^2+9)^{9/2}\,dx$ $x =$ ____________ $dx =$ ____________

9 (12 pts). Find the Partial Fraction Decomposition of $\dfrac{-x^2-4x+9}{(x-2)(x^2-6x+9)}$, but **do not integrate**.

10 (8 pts). Evaluate the indefinite integral. $\int \left(\frac{1}{x^2-8x+17} + \frac{7}{(x-1)^2} - \frac{4}{x+2}\right) dx$

1. (Source: 3.11.31,35) . Use the chain rule within the product rule.

$$\begin{aligned} f'(x) &= \big(\cosh(x^2)\big)'\big(\sinh(x^3)\big) + \big(\cosh(x^2)\big)\big(\sinh(x^3)\big)' \\ &= 2x\sinh(x^2)\sinh(x^3) + 3x^2\cosh(x^2)\cosh(x^3) \end{aligned}$$

2. (Source: 7.1.21) Use integration by parts: $\begin{matrix} u = t & dv = \sinh t\,dt \\ du = dt & v = \cosh t \end{matrix}$ and the indefinite integral becomes

$$\begin{aligned} \int t\sinh t\,dt = \int u\,dv = uv - \int v\,du \\ = t\cosh t - \int \cosh t\,dt = t\cosh t - \sinh t + C. \end{aligned}$$

To evaluate the definite integral, use the Fundamental Theorem of Calculus (don't forget that $\sinh 0 = 0$): $\int_0^2 t\sinh t\,dt = (t\cosh t - \sinh t)\Big|_0^2 = 2\cosh 2 - \sinh 2$

3a. (Source: 4.4.8) $\lim_{x\to 1}\frac{x^{2/3}-1}{x^{5/3}-1}$ has the form $\frac{0}{0}$, so try l'Hospital's Rule. Look instead at the simpler limit $\lim_{x\to 1}\frac{\frac{2}{3}x^{-1/3}}{\frac{5}{3}x^{2/3}} = 2/5$. Therefore, $\lim_{x\to 1}\frac{x^{2/3}-1}{x^{5/3}-1} = 2/5$ also, by l'Hospital's Rule.

3b. (Source: 4.4.11,36) $\lim_{x\to 0}\frac{e^x-e^{-x}-3x}{x-\sin x}$ has the form $\frac{0}{0}$, so try l'Hospital's Rule. Look instead at $\lim_{x\to 0}\frac{e^x+e^{-x}-3}{1-\cos x}$ which "equals" $\frac{-1}{0}$, telling us that the only possible limit is $\pm\infty$. To decide, determine the sign of $\frac{e^x+e^{-x}-3}{1-\cos x}$ for x near 0. Since the top goes to -1, it must be negative for x sufficiently close to zero. The denominator is ≥ 0, since $\cos x$ is always ≤ 1. We conclude that the fraction is negative and blowing up, hence $\lim_{x\to 0}\frac{e^x+e^{-x}-3}{1-\cos x} = -\infty$. By l'Hospital's Rule, $\lim_{x\to 0}\frac{e^x-e^{-x}-3x}{x-\sin x} = -\infty$ also.

4. (Source: 7.3.26) Let $t = \tan x$, $dt = \sec^2 x\,dx$. Then

$$\begin{aligned} \int \tan^4 x\sec^4 x\,dx = \int \tan^4 x\sec^2 x\sec^2 x\,dx = \int \tan^4 x(1+\tan^2 x)\sec^2 x\,dx \\ = \int t^4(1+t^2)\,dt = \int (t^4+t^6)\,dt = \frac{1}{5}t^5 + \frac{1}{7}t^7 + C = \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C \end{aligned}$$

5. (Source: 7.2.45) Use Euler's formula to rewrite the integrand as a sum of sinusoidals:

$$\begin{aligned} \sin(4x)\cos(3x) &= \left(\frac{e^{i4x}-e^{-i4x}}{2i}\right)\left(\frac{e^{i3x}+e^{-i3x}}{2}\right) = \frac{1}{2\cdot 2i}(e^{i7x}-e^{-i7x}+e^{ix}-e^{-ix}) \\ &= \frac{1}{2}\left(\frac{e^{i7x}-e^{-i7x}}{2i} + \frac{e^{ix}-e^{-ix}}{2i}\right) = \frac{1}{2}(\sin(7x)+\sin(x)). \end{aligned}$$

Now integration is straightforward: $\int \frac{1}{2}(\sin(7x)+\sin(x))\,dx = -\frac{1}{2}(\frac{1}{7}\cos(7x)+\cos(x))+C$.

You could instead use the sum and difference formulas for sine:

$$\begin{aligned}\sin(4x+3x) &= \sin(4x)\cos(3x)+\cos(4x)\sin(3x)\\ \sin(4x-3x) &= \sin(4x)\cos(3x)-\cos(4x)\sin(3x)\end{aligned}$$

Add these and divide by 2 to obtain $\frac{1}{2}(\sin(7x)+\sin x) = \sin(4x)\cos(3x)$, and now integrate as above.

Finally, it's possible to solve this problem by using integration by parts twice, but it's much more work and prone to mistakes. Let $u = \sin(4x)$ the first time, and then $u = \cos(4x)$ the second. The original integral then reappears and you can solve for it.

6. (Source: Euler.9.g) Use Euler's formula.

$$\sin^2 x \cos^4 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4.$$

Expand using the second and fourth rows of Pascal's triangle, 1 2 1 and 1 4 6 4 1:

$$\begin{aligned}
&\frac{-1}{2^2}(e^{i2x}-2e^0+e^{-i2x})\frac{1}{2^4}(e^{i4x}+4e^{i2x}+6e^0+4e^{-i2x}+e^{-i4x})\\
=&\frac{-1}{2^6}\left(\begin{array}{ccccccc} & & e^{i2x} & +4e^0 & +6e^{-i2x} & +4e^{-i4x} & +e^{-i6x}\\ & -2e^{i4x} & -8e^{i2x} & -12e^0 & -8e^{-i2x} & -2e^{-i4x} & \\ e^{i6x} & +4e^{i4x} & +6e^{i2x} & +4e^0 & +e^{-i2x} & & \end{array}\right)\\
=&\frac{-1}{2^6}\ (\,e^{i6x}\ \ +2e^{i4x}\ \ -e^{i2x}\ \ -4e^0\ \ -e^{-i2x}\ \ +2e^{-i4x}\ \ +e^{-i6x}\,)\\
=&\frac{-1}{2^6}(e^{i6x}+e^{-i6x}+2e^{i4x}+2e^{-i4x}-e^{i2x}-e^{-i2x}-4)\\
=&\frac{-1}{2^5}(\cos(6x)+2\cos(4x)-\cos(2x)-2)
\end{aligned}$$

and we're done.

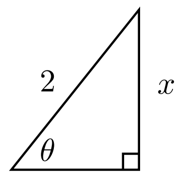
It really helps to maintain columns for the different powers of e in the product, just it helps to maintain columns for different powers of x when multiplying polynomials. Here's the product again (ignoring the -2^{-6}) with just the necessary numbers. Note the resemblance to multiplication base ten.

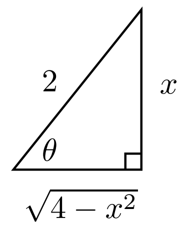
$$\begin{array}{ccccccc}
 & 1 & 4 & 6 & 4 & 1 & \\
 & & 1 & -2 & 1 & & \\ \hline
 & & 1 & 4 & 6 & 4 & 1\\
 & -2 & -8 & -12 & -8 & -2 & \\
1 & 4 & 6 & 4 & 1 & & \\ \hline
1 & 2 & -1 & -4 & -1 & 2 & 1
\end{array}$$

7. (Source: 7.3.1) Evaluate the indefinite integral. $\int \frac{dx}{x^2\sqrt{4-x^2}}$ We want $4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$, so we let $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$. Integral becomes

$$\int \frac{2\cos\theta\, d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}} = \int \frac{2\cos\theta\, d\theta}{(4\sin^2\theta)(2\cos\theta)} = \int \frac{d\theta}{4\sin^2\theta} = \int \frac{1}{4}\csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C.$$

To rewrite this in terms of x, draw a triangle showing the sine of θ equal $x/2$, and then find the third side by Pythagorus:





Therefore $-\frac{1}{4}\cot\theta + C = -\frac{\sqrt{4-x^2}}{4x} + C$.

8a. (Source: 7.3.1,17) $x^2 - 3 = 3\sec^2\theta - 3$, if we let $x = \sqrt{3}\sec\theta$. Then $dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$.

8b. (Source: 7.3.3) Want $x^2 + 9 = 9\tan^2\theta + 9$, so let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$.

9. (Source: 7.4.12,19) Begin by factoring the denominator completely.

$$\frac{-x^2 - 4x + 9}{(x-2)(x^2 - 6x + 9)} = \frac{-x^2 - 4x + 9}{(x-2)(x-3)^2} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

Multiply both sides by $(x-2)(x-3)^2$:

$$-x^2 - 4x + 9 = A(x-3)^2 + B(x-2)(x-3) + C(x-2)$$

Now let $x = 2$, 3, and any other value, say, 0:

$$x = 2 \Rightarrow -3 = A(1) \Rightarrow A = -3$$
$$x = 3 \Rightarrow -12 = C(1) \Rightarrow C = -12$$
$$x = 0 \Rightarrow 9 = 9A + 6B - 2C = 6B - 3 \Rightarrow B = 2$$

You could also have equated the x^2 coefficients to find B: $-1 = A + B$, which again implies $B = 2$. The PDF therefore is

$$\frac{-x^2 - 4x + 9}{(x-2)(x-3)^2)} = -\frac{3}{x-2} + \frac{2}{x-3} - \frac{12}{(x-3)^2}$$

10. (Source: 7.3.19,29) Complete the square in the first integrand $x^2 - 8x + 17 = x^2 - 8x + 16 + 1 = (x-4)^2 + 1$, and the integral becomes $\int \left(\frac{1}{(x-4)^2+1} + 7(x-1)^{-2} - \frac{4}{x+2}\right) dx = \tan^{-1}(x-4) - 7(x-1)^{-1} - 4\ln|x+2| + C$.

(If you like, substitute $u = x - 4$ in the first integral, $v = x - 1$ in the second, and w $= x + 2$ in the third. Then the integral turns into $\int \frac{du}{u^2+1} + \int 7v^{-2}\, dv - 4\int \frac{dw}{w} = \tan^{-1} u + -7v^{-1} - 4\ln|w| + C$, etc..)