1a (16 pts). The graph of the polar equation \( r = 2 + \cos \theta \) appears at the right. Find the total area enclosed by the curve.

1b (6 pts). Find the length of the curve in Problem 1a. Express your answer as a definite integral, but do not evaluate.

2a (5 pts). Find \( \frac{dy}{dx} \) along the curve given by the parametric equations \( x = \sin t, \ y = \sin 2t \). Express your answer in terms of \( t \).

2b (10 pts). Find the times \( t \) between 0 and \( 2\pi \) when the line tangent to the curve in Problem 2a is horizontal and when it is vertical. Label your answers so I can tell which is which. You are not required to find \( x \) and \( y \) at these times.

3 (10 pts). Evaluate the indefinite integral. \( \int \tan x \sec^4 x \, dx \)

4 (12 pts). Evaluate the improper integral, if it converges. \( \int_0^\infty \frac{e^x}{1 + e^{2x}} \, dx \)

5 (15 pts). Let \( R \) denote the region in the \( xy \)-plane bounded by the curves \( y = e^{-x}, \ y = 2, \) and \( x = 1. \)

a. Find the volume of the solid obtained by rotating \( R \) about \( y = 0 \). Express your answer as a definite integral, but do not evaluate.

b. Find the volume of the solid obtained by rotating \( R \) about \( x = 4 \). Express your answer as a definite integral, but do not evaluate.
6 (5 pts). Evaluate the limit. \( \lim_{n \to \infty} \sin \left( \frac{2n\pi}{1 + 8n} \right) \)

7 (6 pts). Determine whether the series converges or diverges. \( \sum_{n=0}^{\infty} \sin \left( \frac{2n\pi}{1 + 8n} \right) \)

(Supporting work not required for Problem 8. Correct answers will receive full credit.)

8 (11 pts). Give a power series representation of the function.
   a. \( \tan^{-1} x \)
   b. \( e^x \)

9 (13 pts). Determine whether the series converges or diverges. \( \sum_{n=1}^{\infty} \frac{\sqrt{n} - 1}{2n^3 + n + 3} \)

10 (13 pts). Find the radius of convergence of the power series. \( \sum_{n=0}^{\infty} \frac{n}{3^n} (x + 2)^n \)

11 (11 pts). Find the Taylor series centered at \( a = 1 \) of the function \( x^3 + 2x^2 - 3 \).

12 (14 pts). Find a power series representation of the function. \( \frac{x^3}{(x + 2)^2} \)

13a (11 pts). Evaluate the indefinite integral. \( \int x \sin 2x \, dx \)

13b (8 pts). Find the average value of the function \( x \sin 2x \) on the interval \([0, \pi]\).

14 (19 pts). Evaluate the indefinite integral. \( \int \frac{dx}{x^2 \sqrt{9 - x^2}} \)

15 (15 pts). Find the partial fraction decomposition of the function. \( \frac{-x^2 + 7x - 2}{(x + 1)^2(x - 1)} \)

Do not integrate the result.
1a. (Source: 10.4.12) Start with $dA = \frac{1}{2} r^2 d\theta$:

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (2 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \cos \theta + \cos^2 \theta) \, d\theta$$

To integrate $\cos^2 \theta$, rewrite it either with Euler’s formula or the double angle formula. Here’s Euler:

$$\cos^2 \theta = \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^2 = \frac{1}{4} (e^{i2\theta} + 2 + e^{-i2\theta}) = \frac{1}{2} \left( \frac{e^{i2\theta} + e^{-i2\theta}}{2} + 1 \right) = \frac{1}{2} (\cos 2\theta + 1),$$

so the area is

$$\frac{1}{2} \int_0^{2\pi} \left( 4 + 4 \cos \theta + \frac{1}{2} (\cos 2\theta + 1) \right) \, d\theta = \frac{1}{2} \int_0^{2\pi} \left( \frac{9}{2} + 4 \cos \theta + \frac{1}{2} \cos 2\theta \right) \, d\theta$$

$$= \frac{1}{2} \left[ \frac{9}{2} \theta + 4 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{1}{2} \left[ \frac{9}{2} \cdot 2\pi \right] = \frac{9\pi}{2}.$$

1b. (Source: 10.4.46) Length is $\int ds = \int \sqrt{r^2 + \left( \frac{dy}{dt} \right)^2} \, d\theta = \int_0^{2\pi} \sqrt{(2 + \cos \theta)^2 + \sin^2 \theta} \, d\theta$, or $\int_0^{2\pi} \sqrt{4 + 4 \cos \theta + \cos^2 \theta + \sin^2 \theta} \, d\theta = \int_0^{2\pi} \sqrt{5 + 4 \cos \theta} \, d\theta$.

2a. (Source: 10.2.16) $\frac{du}{dx} = \frac{dy}{dt} = \frac{2 \cos 2t}{\cos t}$

2b. (Source: 10.2.19) Solve $dx/dt = 0$ and $dy/dt = 0$. We’re looking for solutions $t$ between 0 and $2\pi$, so that $2t$ is between 0 and $4\pi$.

$$\frac{dx}{dt} = \cos t = 0 \quad \Rightarrow \quad t = \pi, \quad \frac{3\pi}{2}.$$  

$$\frac{dy}{dt} = 2 \cos 2t = 0 \quad \Rightarrow \quad 2t = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad \frac{7\pi}{2}.$$  

$$t = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}.$$

The tangent line is horizontal at $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}$, since $dy/dt = 0 \neq dx/dt$ at these times.

The tangent line is vertical at $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, since $dx/dt = 0 \neq dy/dt$ at these times.

3. (Source: 7.2.21, 27) There are at least three ways to solve this problem.

Solution one: $\int \tan x \sec^4 x \, dx = \int \tan x \sec^2 x \sec^2 x \, dx = \int \tan x (\tan^2 x + 1) \sec^2 x \, dx$.

Let $t = \tan x$, $dt = \sec^2 x \, dx$, and integral becomes $\int t(t^2 + 1) \, dt = \int (t^3 + t) \, dt = \frac{t^4}{4} + \frac{t^2}{2} + C = \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + C$.

Solution two: $\int \sec x \, dx = \int \sec x \, dx$. Let $s = \sec x$, $ds = \sec x \tan x \, dx$, so integral becomes $\int s^3 \, ds = \frac{1}{4} s^4 + C = \frac{1}{4} \sec^4 x + C$.

Solution three: $\int \tan x \sec^4 x \, dx = \int \left( \frac{\sin x}{\cos x} \right) \left( \frac{1}{\cos^2 x} \right) \, dx = \int \sin x \cos^{-5} x \, dx$. Let $u = \cos x$, $du = -\sin x \, dx$, so integral becomes $-\int u^{-5} \, du = -\frac{1}{4} u^{-4} + C = -\frac{1}{4} \cos^{-4} x + C$.

4. (Source: 7.8.24) Substitute $u = e^x$, $du = e^x \, dx$. Change the limits as well: $x = 0 \Rightarrow u = 1$ and $x \to \infty \Rightarrow u \to \infty$. The integral becomes

$$\int_1^\infty \frac{du}{u^2 + 1} = \lim_{B \to \infty} \int_1^B \frac{du}{u^2 + 1} = \lim_{B \to \infty} (\tan^{-1} B - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$
5. (Source: 6.2.12, 6.3.17) You don’t need a highly accurate graph of \( R \) to answer this question. \( y = e^{-x} \) is decreasing. \( y = 2 \) and \( x = 1 \) are horizontal and vertical lines, respectively. \( R \) must look something like this (after solving for intersection points and slicing vertically):

\[ (-\ln 2, 2), (1, 2), (1, e^{-1}) \]

When we rotate each rectangle about the horizontal line \( y = 0 \), the result is a washer. Rotating about the vertical line \( x = 4 \) gives a shell.

a. \( V = \int dV = \int_{-\ln 2}^{1} \pi \left( 2^2 - (e^{-x})^2 \right) dx \) (or, \( \int_{-\ln 2}^{1} \pi (4 - e^{-2x}) dx \))

b. \( V = \int dV = \int_{-\ln 2}^{1} 2\pi (4 - x)(2 - e^{-x}) dx \)

6. (Source: 11.1.23) \( \lim_{n \to \infty} \left( \frac{2n\pi}{1+8n} \right) = \frac{\infty}{\infty} \). l’Hospitize: \( \lim_{n \to \infty} \left( \frac{2n\pi}{1+8n} \right) = \frac{\pi}{4} \). Since sin is a continuous function, \( \lim_{n \to \infty} \sin \left( \frac{2n\pi}{1+8n} \right) = \sin \left( \lim_{n \to \infty} \frac{2n\pi}{1+8n} \right) = \sin \left( \frac{\pi}{4} \right) = \sqrt{2}/2 \).

7. (Source: 11.2.29) Since \( \lim_{n \to \infty} \sin \left( \frac{2n\pi}{1+8n} \right) \) is not zero, the series \( \sum_{n=0}^{\infty} \sin \left( \frac{2n\pi}{1+8n} \right) \) must diverge by the \( n \)th Term Test (a.k.a. the Test for Divergence).

8. (Source: 11.9.18, 11.10.31) You can either rederive these two series or memorize the results (or some combination of the two). Here’s how you can remember the two.

a. \( \tan^{-1} x = \int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} (-x^2)^n dx = \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \).
   (The constant of integration turns out to be zero.)

b. In general, the Maclaurin series is \( \sum_{n=0}^{\infty} f^{(n)}(0)x^n/n! \), but when \( f(x) = e^x \), \( f^{(n)}(0) = e^0 = 1 \) for all \( n \). Therefore the series is \( \sum_{n=0}^{\infty} x^n/n! \).
9. (Source: 11.4.21) This series probably behaves the same as \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^3 + n + 3} \), which is (one-half times) a convergent \( p \)-series, since \( p = 2.5 > 1 \). Because

\[
0 \leq \frac{\sqrt{n} - 1}{2n^3 + n + 3} \leq \frac{\sqrt{n}}{2n^3},
\]

the Comparison Test implies that \( \sum_{n=1}^{\infty} \frac{\sqrt{n} - 1}{2n^3 + n + 3} \) must also converge.

You could also use the Limit Comparison Test:

\[
\lim_{n \to \infty} \frac{\frac{\sqrt{n} - 1}{2n^3 + n + 3}}{\frac{1}{n^{5/2}}} = \lim_{n \to \infty} \frac{(\sqrt{n} - 1)2n^3}{\sqrt{n}(2n^3 + n + 3)} = \lim_{n \to \infty} \left( \frac{\sqrt{n} - 1}{\sqrt{n}} \right) \left( \frac{2n^3}{2n^3 + n + 3} \right) = 1 \cdot 1 = 1.
\]

Since this limit is positive and finite, and \( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \) converges, so does \( \sum_{n=1}^{\infty} \frac{\sqrt{n} - 1}{2n^3 + n + 3} \), by the LCT.

10. (Source: 11.8.18) Solution one. Apply the Root test. Let’s let \( \rho = \lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \left( \frac{n}{3^n} |x + 2|^n \right)^{1/n} = \lim_{n \to \infty} \frac{n^{1/n}}{3} |x + 2| \).

Because \( \lim_{n \to \infty} n^{1/n} = 1 \) (a Famous Limit Everybody Should Know), \( \rho = \frac{1}{3} |x + 2| \). The series converges absolutely when \( \frac{1}{3} |x + 2| < 1 \), or \( |x + 2| < 3 \), so the radius of convergence is 3.

Solution two. Apply the Ratio test. Let \( \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n+1}{n} \frac{(x + 2)^{n+1}}{3^n (x + 2)^n} = \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{|x + 2|^{n+1}}{(x + 2)^n} \]

\[
= \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{1}{3} |x + 2| = \frac{1}{3} |x + 2| \]

The remaining steps are the same as in Solution one.

You were not required to find the endpoints of the interval of convergence, but

\[
|x + 2| < 3 \implies -3 < x + 2 < 3 \implies -5 < x < 1,
\]

so the endpoints are -5 and 1.
11. (Source: 11.10.13) Here’s a table of the coefficients of the Taylor series.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(1)/n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x^3 + 2x^2 - 3$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$3x^2 + 4x$</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>$6x + 4$</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$\geq 4$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Since $f^{(n)}(x) = 0$ for $n \geq 4$ are zero, the Taylor series is finite (and converges to $f(x)$):

$$f(x) = 7(x - 1) + 5(x - 1)^2 + (x - 1)^3.$$  

12. (Source: 11.9.17) Solution one. As in section 11.9, we’ll obtain the desired series from the geometric series. First,

$$\frac{1}{x+2} = \frac{1}{2} \left( \frac{1}{\frac{x}{2} + 1} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}.$$  

Now

$$\frac{1}{(x+2)^2} = - \frac{d}{dx} \left( \frac{1}{x+2} \right) = - \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}$$

$$= - \sum_{n=0}^{\infty} (-1)^n \frac{nx^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{nx^{n-1}}{2^{n+1}}.$$  

Finally,

$$\frac{x^3}{(x+2)^2} = x^3 \left( \frac{1}{(x+2)^2} \right) = x^3 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{nx^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{nx^{n+2}}{2^{n+1}}.$$  

Solution two uses the Binomial Series $(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$:

$$\frac{x^3}{(x+2)^2} = \frac{x^3}{4} \left( \frac{x}{2} + 1 \right)^{-2} = \frac{x^3}{4} \sum_{n=0}^{\infty} \binom{-2}{n} \left( \frac{x}{2} \right)^n = \frac{x^3}{4} \sum_{n=0}^{\infty} \binom{-2}{n} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \binom{-2}{n} \frac{x^{n+3}}{2^{n+2}}.$$  

13a. (Source: 7.1.6) Use integration by parts.

$$u = x \quad dv = \sin 2x \, dx$$

$$du = dx \quad v = -\frac{1}{2} \cos 2x$$
\[ \int x \sin 2x \, dx = \int u \, dv = uv - \int v \, du \]
\[ = -\frac{1}{2} x \cos 2x + \int \frac{1}{2} \cos 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C \]

13b. (Source: 6.5.7) By definition, the average value is
\[
\frac{1}{\pi - 0} \int_0^\pi x \sin 2x \, dx = \frac{1}{\pi} \left( -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right) \bigg|_0^\pi = \frac{1}{\pi} \left( -\frac{1}{2} \pi \cos 2\pi + \frac{1}{4} \sin 2\pi \right) - \left( -\frac{1}{2} \cdot 0 \cos 0 + \frac{1}{4} \sin 0 \right) = -\frac{1}{2}.
\]

14. (Source: 7.3.7) Trigonometric substitution. We would like \(9 - x^2 = 9 - 9 \sin^2 \theta = 9 \cos^2 \theta\), so we let \(x = 3 \sin \theta\) and \(dx = 3 \cos \theta \, d\theta\). Then
\[
\int \frac{dx}{x^2 \sqrt{9 - x^2}} = \int \frac{3 \cos \theta \, d\theta}{9 \sin^2 \theta \sqrt{9 \cos^2 \theta}} = \int \frac{3 \cos \theta \, d\theta}{9 \sin^2 \theta} = \frac{1}{9} \int \csc^2 \theta \, d\theta = -\frac{1}{9} \cot \theta + C
\]
To rewrite this answer in terms of the original variable \(x\), draw a right triangle with interior angle \(\theta\). Label two sides using \(\sin \theta = x/3\), and then find the third side by the Pythagorean theorem:

So our integral is
\[
-\frac{1}{9} \cot \theta + C = -\frac{1}{9} \frac{\sqrt{9 - x^2}}{x} + C
\]

15. (Source: 7.4.20) Note that the degree of the numerator is less than that of the denominator, so long division is unnecessary. The PFD has the form
\[
\frac{-x^2 + 7x - 2}{(x + 1)^2(x - 1)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x - 1}.
\]
Multiply both sides by \((x + 1)^2(x - 1)\) to obtain
\[
-x^2 + 7x - 2 = A(x + 1)(x - 1) + B(x - 1) + C(x + 1)^2.
\]
We’ll solve for the constants by substituting values of \(x\) into both sides.
\[
\begin{align*}
x &= 1 : & 4 &= 4C & \implies C &= 1 \\
x &= -1 : & -10 &= -2B & \implies B &= 5 \\
x &= 0 : & -2 &= - A - B + C = - A - 5 + 1 & \implies A &= -2
\end{align*}
\]
So the partial fraction decomposition is
\[
\frac{-x^2 + 7x - 2}{(x + 1)^2(x - 1)} = \frac{-2}{x + 1} + \frac{5}{(x + 1)^2} + \frac{1}{x - 1}.
\]