1 (14 pts). Determine whether the series converges or diverges. \[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}} \]

2 (8 pts). Find the sum of the series, if it converges: \[ 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \cdots \]

3 (9 pts). Evaluate the limit, or show that it does not exist.

   a. \[ \lim_{n \to \infty} \frac{4n^2 - 3}{-n^2 + n} \]
   b. \[ \lim_{n \to \infty} \left( \frac{4n^2 - 3}{-n^2 + n} \right) (-1)^n \]

4a (9 pts). Find a formula for the \( n \)th partial sum of \[ \sum_{n=1}^{\infty} \left( \sin \left( \frac{1}{n} \right) - \sin \left( \frac{1}{n+1} \right) \right) \]

4b (5 pts). Find the sum of the series in 4a, or show that the series diverges.

5 (16 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. \[ \sum_{n=1}^{\infty} (-1)^n \frac{n-2}{n^2} \]

6 (10 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. \[ \sum_{n=0}^{\infty} (-1)^{n-1} \frac{n^2}{e^n} \]

7 (12 pts). Find the radius of convergence of the power series \[ \sum_{n=0}^{\infty} \frac{(x + 1)^n}{n!} \]

8. Fact: The radius of convergence of the power series \[ \sum_{n=0}^{\infty} n(x - 2)^n \] is 1.

   a (4 pts). What are the endpoints of the interval of convergence of this series?
   b (13 pts). Determine which endpoints, if any, belong to the interval of convergence.
1. (Source: 11.3.21, 27, 11.7.7) Try the Integral Test. Observe that \( f(x) = \frac{1}{x(\ln x)^{2/3}} \) is positive when \( x > 1 \) and decreasing, since it’s the reciprocal of the positive and increasing function \( x(\ln x)^{2/3} \). Therefore, the Integral Test implies that the series \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}} \) and the improper integral \( \int_{2}^{\infty} \frac{dx}{x(\ln x)^{2/3}} \) either both converge or both diverge.

Rewrite this integral with the substitution \( v = \ln x \), \( dv = x^{-1} \, dx \). Note that the endpoints \( x = 2 \) and \( x = \infty \) change to \( u = \ln 2 \) and \( u = \infty \).

\[
\int_{\ln 2}^{\infty} \frac{dv}{v^{2/3}} = \lim_{\omega \to \infty} \int_{\ln 2}^{\omega} v^{-2/3} \, dv = \lim_{\omega \to \infty} 3v^{1/3}\bigg|_{\ln 2}^{\omega} = \lim_{\omega \to \infty} 3(\omega^{1/3} - (\ln 2)^{1/3}) = \infty.
\]

Since the integral diverges, so does the series.

2. (Source: 11.2.11) Every term is \( r = -\frac{2}{3} \) times the previous term, so the series is geometric. Because \( |r| \leq 1 \), the series converges. \( a \) is the 1, the first term of the series, and the sum is \( \frac{a}{1-r} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5} \).

3a. (Source: 11.1.19,26) The limit as \( n \to \infty \) is the same as the limit of the lead terms of the numerator and denominator: \( \lim_{n \to \infty} \frac{4n^2-3}{n^2+n} = \lim_{n \to \infty} \frac{4n^2}{n^2} = -4 \).

3b. We learned in 3a that \( \left(\frac{4n^2-3}{n^2+n}\right)(-1)^n \to -4 \). Therefore the even numbered terms of \( \left(\frac{4n^2-3}{n^2+n}\right)(-1)^n \to -4 \) and the odd numbered terms go to +4. So, \( \lim_{n \to \infty} \left(\frac{4n^2-3}{n^2+n}\right)(-1)^n \) does not exist.

4a. (Source: 11.2.39) The series is telescoping. Here are the first few partial sums..

\[
s_1 = \sin 1 - \sin \left(\frac{1}{2}\right), \quad s_3 = s_2 + \sin \left(\frac{1}{3}\right) - \sin \left(\frac{1}{4}\right) = \sin 1 - \sin \left(\frac{1}{2}\right) + \sin \left(\frac{1}{3}\right) - \sin \left(\frac{1}{4}\right) = \sin 1 - \sin \left(\frac{1}{3}\right), \quad s_4 = s_3 + \sin \left(\frac{1}{4}\right) - \sin \left(\frac{1}{5}\right) = \sin 1 - \sin \left(\frac{1}{4}\right).
\]

In general, \( s_n = \sin 1 - \sin \left(\frac{1}{n+1}\right) \).

4b. The sum of the series is the limit of its partial sums:

\[
\sum_{n=1}^{\infty} \left(\sin \left(\frac{1}{n}\right) - \sin \left(\frac{1}{n+1}\right)\right) = \lim_{n \to \infty} \left(\sin 1 - \sin \left(\frac{1}{n+1}\right)\right) = \sin 1 - \sin 0 = \sin 1.
\]

5. (Source: 11.6.10) Test for absolute convergence first. \( \sum_{n=1}^{\infty} \left| (-1)^n \frac{n-2}{n^2} \right| = \sum_{n=1}^{\infty} \frac{n-2}{n^2} \). Limit-compare this to the simpler series \( \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} \), the divergent Harmonic series.

\[
\lim_{n \to \infty} \frac{n-2}{n} = \lim_{n \to \infty} \frac{(n-2)n}{n^2} = \lim_{n \to \infty} \frac{(1 - \frac{2}{n})n^2}{n} = \lim_{n \to \infty} (1 - \frac{2}{n}) = 1.
\]

Since this limit is finite and positive, and the Harmonic series diverges, \( \sum_{n=1}^{\infty} \frac{n-2}{n^2} \) must also diverge. \( \sum_{n=1}^{\infty} (-1)^n \frac{n-2}{n^2} \) fails to converge absolutely.
Next test for (conditional) convergence. \( b_n = \frac{n-2}{n^2} = \frac{1}{n} - \frac{2}{n^2} \to 0 \) as \( n \to \infty \). To see if it’s decreasing, check the sign of its derivative. 

\[
b'_n = \frac{1 - n^2 - (n-2)2n}{n^4} = \frac{-n^2 + 4n}{n^4} = \frac{4 - n}{n^3} < 0 \quad \text{once} \quad n > 4,
\]

so \( b_n \) is decreasing (eventually), and \( \sum_{n=1}^{\infty} (-1)^n \frac{n-2}{n^2} \) converges by the Alternating Series Test.

Since it converges, but not absolutely, \( \sum_{n=1}^{\infty} (-1)^n \frac{n-2}{n^2} \) converges conditionally.

(We can’t use the Comparison Test in place of the Limit Comparison Test because \( \frac{n-2}{n^2} < \frac{1}{n} \). Root and Ratio tests are inconclusive since corresponding limits equal 1.)

6. (Source: 11.6.2) Try the Root Test. Using the Famous Limit Everyone Should Know,

\[
\lim_{n \to \infty} n^{p/n} = 1 \quad \text{for all numbers} \quad p,
\]

\[
\lim_{n \to \infty} \left( \left| (-1)^{n-1} \frac{n^2}{e^n} \right| \right)^{1/n} = \lim_{n \to \infty} \left( n^2 e^{-n} \right)^{1/n} = \lim_{n \to \infty} n^{2/n} e^{-1} = e^{-1}. \quad \text{This limit is}
\]

less than one, so \( \sum_{n=0}^{\infty} (-1)^{n-1} \frac{n^2}{e^n} \) converges absolutely.

7. (Source: 11.8.7, 15-25) Try the ratio test:

\[
\lim_{n \to \infty} \frac{|x+1|^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{|x+1|^{n+1}}{|x+1|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{|x+1|}{(n+1)} = 0
\]

Since this limit is less than 1, the series converges for all real numbers \( x \). Radius of convergence is \( \infty \).

8a. (Source: 11.8.18) Since the radius of convergence is 1 and the center of the interval of convergence is 2, the endpoints at \( 2 \pm 1 = 1 \) and 3.

8b. Test the convergence of the power series at these two \( x \)-values.

At \( x = 3 \), the series is \( \sum_{n=0}^{\infty} n(3-2)^n = \sum_{n=0}^{\infty} n \). Since \( \lim_{n \to \infty} n = \infty \neq 0 \), this series diverges by the \( n \)th Term Test. \( x = 3 \) does not belong to the interval of convergence.

At \( x = 1 \), the series is \( \sum_{n=0}^{\infty} n(1-2)^n = \sum_{n=0}^{\infty} n(-1)^n \). This series also diverges by the \( n \)th Term Test because \( n(-1)^n \) fails to converge to zero. (The even numbered terms go to \( \infty \) and the odd numbered terms go to \( -\infty \).) \( x = 1 \) does not belong to the interval of convergence.

(The interval of convergence is \( (1, 3) \).)