1 (19 pts). Let \( R \) denote the region in the \( xy \)-plane bounded by the two curves \( y = x^2 - 4x \) and \( y = 6 - x^2 \). Find the volume that results from rotating \( R \) about the given line. Express your answer as a definite integral, but do not evaluate.

Work in the white space provided. Clearly label your answers “a.” and “b.”

a. \( x = -5 \)  

b. \( y = 9 \)

2. Evaluate the limit.

a (9 pts). \( \lim_{x \to 2} \left( \frac{x}{x - 2} - \frac{6}{x^2 - x - 2} \right) \)  
b (13 pts). \( \lim_{x \to 0} (1 + 3x)^{1/x} \)

3 (15 pts). Evaluate the improper integral, if it converges.

a. \( \int_{3}^{\infty} \frac{x^2}{\sqrt{x^3 - 8}} \, dx \)  
b. \( \int_{3}^{2} \frac{x^2}{\sqrt{x^3 - 8}} \, dx \)

4 (2 pts). Does \( \int_{-\pi/2}^{\pi/2} \tan x \, dx \) converge or diverge? (To decide, you might want to use the facts that \( \int_{-\pi/2}^{0} \tan x \, dx = -\infty \) and \( \int_{0}^{\pi/2} \tan x \, dx = \infty \).)

No work necessary on Problem 4. Correct answers will receive full credit.

5 (11 pts). A spring has natural length 0.5 m. If a force of 20 N is required to keep it stretched to a length of 0.7 m, how much work is required to stretch it from 0.6 to 0.9 m? Express your answer as a definite integral, but do not evaluate.

6a (8 pts). Find the average value \( g_{\text{ave}} \) of the function \( g(x) = (x + 1)^3 \) on the interval \([-1, 2]\).

6b (3 pts). This is a continuation of 6a above.

Find a number \( c \) in \([-1, 2]\) for which \( g(c) = g_{\text{ave}} \) or explain why none exists.

7 (13 pts). Find the length of the curve \( y = \frac{1}{4}x^2 - \frac{1}{2} \ln x \) over \( 1 \leq x \leq 2 \).

8 (7 pts). Find the area swept out by the curve in Problem 7 when it is rotated about the given axis. Express your answer as a definite integral, but do not evaluate.

a. About the \( x \)-axis.  
b. About the \( y \)-axis.
(1 pts). (Source: 6.3.18) You don’t need a highly accurate graph of \( R \) to answer this question. \( y = x^2 - 4x \) is a parabola that opens up and \( y = 6 - x^2 \) opens down. \( R \) must look something like this (after solving for intersection points and slicing vertically):

\[
\begin{align*}
(-1, 5) \\
(3, -3)
\end{align*}
\]

When we rotate each rectangle about the vertical line \( x = -5 \), the result is a shell. Rotating about the horizontal line \( y = 9 \) gives a washer.

\[
\begin{align*}
\text{a. } V &= \int dV = \int_{-1}^{3} 2\pi(x + 5)(6 + 4x - 2x^2) \, dx \\
\text{b. } V &= \int dV = \int_{-1}^{3} \pi((9 + 4x - x^2)^2 - (3 + x^2)^2) \, dx
\end{align*}
\]

2a. (Source: 4.4.47) Add the fractions, then either use l’Hospital’s Rule, or simply factor and cancel the common factor.

\[
\lim_{x \to 2} \left( \frac{x}{x - 2} - \frac{6}{(x + 1)(x - 2)} \right) = \lim_{x \to 2} \frac{x(x + 1) - 6}{(x + 1)(x - 2)} = \lim_{x \to 2} \frac{x^2 + x - 6}{(x + 1)(x - 2)} = \lim_{x \to 2} \frac{(x + 3)(x - 2)}{(x + 1)(x - 2)} = \lim_{x \to 2} \frac{x + 3}{x + 1} = \frac{5}{3}
\]

2b. (Source: 4.4.55) \((1 + 3x)^{1/x} \to 1^\infty\), a indeterminate form. Let \( y = (1 + 3x)^{1/x} \). Then \( \ln y = \frac{\ln(1 + 3x)}{x} \to 0 \). Apply l’Hospital’s Rule: \( \lim_{x \to 0} \frac{3x}{x} = 3 \). Therefore \( \ln y \) also \( \to 3 \), and so \( y = e^{\ln y} \to e^3 \). That is, \( \lim_{x \to 0} (1 + 3x)^{1/x} = e^3 \).

3a. (Source: 7.8.23) Make the substitution \( u = x^3 - 8 \), \( du = 3x^2 \, dx \), change the limits \( (x = 3, x \to \infty) \) becomes \( u = 27 - 8 = 19, u \to \infty \) and the improper integral becomes

\[
\frac{1}{3} \int_{19}^{\infty} \frac{1}{u^{1/2}} \, du = \lim_{B \to \infty} \frac{1}{3} \int_{19}^{B} \frac{1}{u^{1/2}} \, du = \lim_{B \to \infty} \frac{2}{3} \sqrt{u}_{19}^{B} = \lim_{B \to \infty} \frac{2}{3}(\sqrt{B} - \sqrt{19}) = \infty.
\]
(You could also conclude that this integral diverges because first, \( \int_1^\infty \frac{1}{u^{1/2}} \, du \) is a \( p \)-integral with \( p = 1/2 < 1 \) and therefore diverges; second, a constant factor \( \frac{1}{3} \) does change the convergence or divergence of an improper integral; and third, the difference between \( \int_1^\infty \frac{1}{u^{1/2}} \, du \) and \( \int_1^{19} \frac{1}{u^{1/2}} \, du \) is the (finite) number \( \int_1^{19} \frac{1}{u^{1/2}} \, du \).

3b. This integral is improper because the integrand blows up at \( x = 2 \). Under the same substitution as in part a, the integral becomes

\[
\frac{1}{3} \int_2^3 \frac{3x^2}{\sqrt{x^3 - 8}} \, dx = \frac{1}{3} \int_0^{19} \frac{1}{u^{1/2}} \, du = \lim_{B \to 0^+} \frac{1}{3} \int_B^{19} u^{-1/2} \, du = \frac{2}{3} \left( \frac{u^{1/2}}{B} \right) \bigg|_B^{19} = \lim_{B \to 0^+} \frac{2}{3} \left( \sqrt{19} - \sqrt{B} \right) = \frac{2\sqrt{19}}{3}.
\]

4. (Source: 7.8.31) \( \int_{-\pi/2}^{\pi/2} \tan x \, dx \) diverges. In order for it to converge, both \( \int_{-\pi/2}^{\pi/2} \tan x \, dx \) and \( \int_0^{\pi/2} \tan x \, dx \) are required to converge.

5. (Source: 6.4.8) Hooke’s Law states that force \( f(x) \) is proportional to the resulting displacement \( x \) from the spring’s natural length. \( f(0.2) = 20 = k \cdot 0.2 \implies k = 20/0.2 = 100 \). Let \( dw \) be the work to move the end of the spring \( dx \) m when the spring is displaced \( x \) m. Over this infinitesimal distance, force is constant, so \( dw = (\text{magnitude of force}) (\text{distance}) = 100x \, dx \). To stretch the spring from 0.6 to 0.9 m means to displace it from 0.1 to 0.4 m beyond its natural length. Total work is \( w = \int dw = \int_{0.1}^{0.4} 100x \, dx \).

If you instead let \( y \) be the length of the spring, then work is \( \int_{0.6}^{0.9} 100(y - 0.5) \, dy \).

6a. (Source: 6.5.9) \( g_{\text{ave}} = \frac{1}{2 - (-1)} \int_{-1}^{2} (x + 1)^3 \, dx = \frac{1}{12} (x + 1)^4 \bigg|_{-1}^{2} = \frac{3^4 - 2^4}{12} = \frac{27}{4} \).

6b. Such a \( c \) is guaranteed by the Mean Value Theorem for Integrals. To find \( c \), take the cube root:

\[
(c + 1)^{3/4} = \frac{27}{4} \implies c + 1 = \frac{3}{4} \implies c = \frac{3^{4/3} - 1}{4}.
\]

\( c = -1 + \sqrt[4]{82/12} \) also received full credit. Here’s a graph of \( g(x) = y \), \( y = 27/4 \) and \( x = c \) on \([-1, 2]\).

7. (Source: 8.2.16) \( s = \int ds = \int_1^2 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_1^2 \sqrt{1 + \left( \frac{1}{2} x - \frac{1}{2} x^{-1} \right)^2} \, dx = \int_1^2 \sqrt{\frac{1}{4} x^2 + \frac{1}{2} + \frac{1}{4} x^{-2}} \, dx = \int_1^2 \left( \frac{1}{2} x + \frac{1}{2} x^{-1} \right) \, dx = \left( \frac{1}{4} x^2 - \frac{1}{2} \ln x \right) \bigg|_1^2 = \frac{3}{4} - \frac{1}{2} \ln 2 \).

8a. (Source: 8.2.16) \( A = \int dy \cdot ds = \int_0^2 2\pi y \, ds = \int_1^2 2\pi \left( \frac{1}{4} x^2 - \frac{1}{2} \ln x \right) \left( \frac{1}{2} x + \frac{1}{2} x^{-1} \right) \, dx \)

8b. \( A = \int dA = \int ds \cdot \int 2\pi x \, ds = \int_1^2 2\pi x \left( \frac{1}{2} x + \frac{1}{2} x^{-1} \right) \, dx \)

No penalty in Problem 8 if you correctly used the \( ds \) you found in Problem 7.