How to write subspace proofs

Problem: H is a subset of a known vector space V. Is H a subspace of V?

Here are some guidelines and, in case the answer is yes, a template for a proof.

1. Know V and H. It's impossible to proceed on this problem without understanding what exactly is meant by V and H. Look for V discussed as an example in your text or notes from class. Don't continue until you can write down a specific example of a vector in V. (Since V is a vector space, each element of V is called a "vector," even if V isn't $\mathbb{R}^n$ for some n.)

H is probably defined by a rule, which I'll call the H-rule, and you should be able to write down a specific example of an element of H. For example,

$$H = \{x \in M_{2\times 2} \mid x_{1,1} = x_{2,2}\}$$

says that H is the collection of all 2×2 matrices that satisfy the H-rule $x_{1,1} = x_{2,2}$. The vectors $\begin{pmatrix} 1 & 5 \\ -3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ belong to H.

2. $\mathbf{0} \in H$? Is the zero vector $\mathbf{0}$ in V also a member of H? First think what exactly is meant by the symbol $\mathbf{0}$ in the vector space V. In $M_{2\times 2}$, the symbol $\mathbf{0}$ stands for the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. In $C[a, b]$, the vector space of all functions $f : [a, b] \to \mathbb{R}$ that are continuous on the interval $[a, b]$, the symbol $\mathbf{0}$ stands for the constant function whose output is the number 0 for every input: $a \le t \le b \implies \mathbf{0}(t) = 0$. Your proof should begin with the sentence

> In the vector space V, the zero vector $\mathbf{0}$ is ⟨`explicit description of` $\mathbf{0}$ `here`⟩.

Think. Does the $\mathbf{0}$ in V satisfy the H-rule? Write a sentence explaining why it does or does not. Be specific. $\mathbf{0}$ is one particular member of V. Either continue your proof with the sentence

> Since ⟨`explain that` $\mathbf{0}$ `satisfies` H`-rule`⟩, $\mathbf{0}$ is an element of H.

or finish it with the sentence

> Since ⟨`explain that` $\mathbf{0}$ `violates` H`-rule`⟩, $\mathbf{0}$ is not in H, and H is not a subspace of V.

3. H closed under vector addition? The only reason to proceed to this step is that you've already proven that $\mathbf{0} \in H$. It would probably be helpful for you to come up with some specific examples of u and v in H and check to see if $u+v$ is also in H. Then, **either** give a specific example of 2 elements u in v in H whose sum is not in H (thus proving that H is not a subspace), **or** write some sentences to explain why, if u and v are in H, so must $u + v$. In the second case, your examples will not suffice, so there's really no point in including them in your proof (unless they illustrate some especially difficult point). A proof will go something like the following.

> Suppose that u and v are in H, so that $\langle$`what the` H`-rule says about` $u\rangle$ and $\langle$`what the` H`-rule says about` $v\rangle$. Then $u + v = \langle$`explain here what exactly is` $u{+}v\rangle$ is in H because $\langle$`explanation of why` $u{+}v$ `satisfies the` H`-rule`$\rangle$. Thus H is closed under vector addition.

4. H **closed under scalar multiplication?** By now you've already proven that $\mathbf{0} \in H$ and that H is closed under $+$. Think of some examples of cu for various $u \in H$ and scalars c for insight. **Either** produce one specific example of a vector u in H and scalar c for which cu is not in H, **or** write a proof as follows:

> Suppose that u is in H, so that $\langle$`what the` H`-rule says about` $u\rangle$ and that c is a scalar. Then $cu = \langle$`explain here what exactly is` $cu\rangle$ is in H because $\langle$`explanation of why` cu `satisfies the` H`-rule`$\rangle$. Thus H is closed under scalar multiplication.

5. Write your conclusion.

> Since H is nonempty and closed under vector addition and scalar multiplication, H is a subspace of V.

Notes. Depending on the spaces in question, it may be possible to prove that H is a subspace in other ways, e.g., if H is $\operatorname{Nul} A$ or $\operatorname{Col} A$ for some matrix A or the span of some specific vectors in V.

If you recognize that H is $\ker T$ for some linear transformation $T : V \to W$, or $\operatorname{ran} T$ for some linear transformation $T : W \to V$, then you'll know that H is a subspace, but to use that fact in your proof will require that you explain what T and W are and prove that T is linear. It may be easier simply to prove that H is nonempty and closed under vector addition and scalar multiplication.

Example 1. Let $\mathbb{P}$ denote the set of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$, and let $H = \{p \in \mathbb{P} \mid p(1) = 0\}$. Prove or disprove that H is a subspace of $\mathbb{P}$.

Think of examples: $\mathbb{P}$ *consists of all functions of the form* $p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$ *for some nonnegative integer* n *and real numbers* $a_0, a_1, a_2, \ldots, a_n$*. The set* $\mathbb{P}$ *is itself a well-known example of a vector space, so we know that the sum of two polynomials is a polynomial, and that the product of a scalar and a polynomial is a polynomial. The polynomials* $p(t) = t - 1$ *and* $q(t) = 1 - t^2$ *are specific vectors in* H*.*

Proof: In the vector space $\mathbb{P}$, the zero vector $\mathbf{0}$ is the constant function whose output is 0 for every value of the input. That is, $\mathbf{0}(t) = 0$ for every real number t. Since $\mathbf{0}(1) = 0$, $\mathbf{0}$ is an element of H.

Think of examples: If $p(t) = t - 1$ *and* $q(t) = 1 - t^2$*, then* $p(t) + q(t) = t - t^2$*, which again equals 0 when* $t = 1$*.*

(**proof, continued**)

Suppose that u and v are in H, so that u and v are polynomials and $u(1) = 0$ and $v(1) = 0$. Then $u + v$ is the function defined by the rule $\big(u + v\big)(t) = u(t) + v(t)$. The sum $u + v$ is in H because $\big(u + v\big)(1) = u(1) + v(1) = 0 + 0 = 0$. Thus H is closed under vector addition.

Think of examples: If $p(t) = t - 1$ *and* $c = 55$*, then* $cp(t) = 55t - 55$*, which again equals 0 when* $t = 1$*.*

(**proof, continued**)

Suppose that u is in H, so that $u(1) = 0$, and that c is a scalar. Then cu is the function defined by the rule $(cu)(t) = c \cdot u(t)$. The product cu is in H because $(cu)(1) = c \cdot u(1) = c \cdot 0 = 0$. Thus H is closed under scalar multiplication.

Conclusion: Since H is nonempty and closed under both vector addition and scalar multiplication, H is a subspace of $\mathbb{P}$. ■

Example 2. Let

$$H = \left\{x \in M_{2\times 2} \mid x_{1,1}^2 = x_{2,2}^2\right\}$$

Prove or disprove that H is a subspace of $M_{2\times 2}$.
Think of examples: $M_{2\times 2}$ *is the set of all* 2×2 *matrices, e.g.*

$$\begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 4 \\ -3 & 1 \end{pmatrix} \quad \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$$

Of these, the second and third belong to H*. The zero vector* $\mathbf{0}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies the H*-rule and so belongs to* H*. However, the squaring in the* H*-rule doesn't look very linear, so we should be skeptical that* H *is a subspace. In fact, when we add the two vectors in* H *from our example above, we get something that violates the* H*-rule.*

Proof: The matrix $\begin{pmatrix} 1 & 4 \\ -3 & 1 \end{pmatrix}$ is in H because $x_{1,1}^2 = 1^2 = x_{2,2}^2$. The matrix $\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$ is also in H, because $x_{1,1}^2 = (-3)^2 = 9 = 3^2 = x_{2,2}^2$. However, their sum,

$$\begin{pmatrix} 1 & 4 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -3 & 4 \end{pmatrix}$$

is **not** in H because $x_{1,1}^2 = (-2)^2 = 4 \neq 4^2 = x_{2,2}^2$. Therefore H fails to be closed under vector addition and is not a subspace of $M_{2\times 2}$. ■

One last piece of advice:

Put yourself in your reader's place.

That is, think of how your proof will appear to your reader and strive to make everything clear. It can hardly be called a proof if you're the only person who can understand it.

Does your proof make grammatical sense? Write in English, in sentences and paragraphs. When you use mathematical symbols, make sure that their literal translation into words fits in the current sentence.

Have you explained new symbols as they come up? Before you use a new symbol in a sentence, first make it clear what that symbol means. Notice how the second paragraph to the proof in Example 1 begins

> Suppose that u and v are in H, so that u and v are polynomials and $u(1) = 0$ and $v(1) = 0$.

If this sentence had simply read

> $u(1) = 0$ and $v(1) = 0$.

the reader would naturally wonder what u and v refer to, and whether the author is claiming that $u(1)$ and $v(1)$ are zero for some u and v mentioned earlier, or for some u and v yet to be determined, or for *all* u and v.

Have you reused a symbol from a previous paragraph? Make it clear whether or not the X in the current paragraph means precisely the same as the X that was used earlier.

Some known vector spaces. Here's a list of names and definitions for some vector spaces that appear in exercises.

$\mathbb{R}^n$: The set of all $n \times 1$ matrices.

$M_{m\times n}$: The set of all $m \times n$ matrices.

$\mathbb{S}$: The set of all doubly infinite sequences of numbers:

$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \ldots)$$

$\mathbb{P}$: The set of all polynomials of one variable, that is, functions of the form $p(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$ for some nonnegative integer n and real numbers $a_0, a_1, a_2, \ldots, a_n$.

$\mathbb{P}_n$: The set of all polynomials of one variable of degree less or equal n.

$C[a, b]$: The set of all real-valued functions that are continuous on the closed interval $[a, b]$.

$C^n[a, b]$: The set of all real valued functions that are n-times differentiable on $[a, b]$ and whose n derivative is continuous on $[a, b]$.

Exercises. Prove or disprove that the following sets are subspaces.

1. $\Lambda = \{\, A \in M_{3\times 2} : A_{3,2} = 2 \,\}$
2. $\Omega = \{\, A \in M_{3\times 3} : \sum_{i=1}^{3} A_{i,i} = 0 \,\}$
3. $\Xi = \{\, y \in \mathbb{S} : y_k = 0 \;\forall k < 0 \,\}$
4. $\Psi = \{\, x \in \mathbb{S} : \lim_{k\to\infty} x_k = 0 \,\}$
5. $K = \{\, f \in C[0, 1] : \int_0^1 f(t)\, dt = 0 \,\}$
6. $L = \{\, f \in C^1(-\infty, \infty) : f'(t) - 2f(t) = e^t \,\}$
7. $H = \{\, f \in C[0, 1] : \int_{-1}^1 t^3 f(t)\, dt = 0 \,\}$
8. $K = \{\, f \in C[0, 1] : \int_0^1 f(t)\, dt = 1 \,\}$
9. $M = \{\, f \in C[0, 1] : f(0) = f(1/2) \,\}$
10. $\Upsilon = \{\, p \in \mathbb{P} : p(t + 1) = p(t) \,\}$