MATH 203–01&03 (Kunkle), Quiz 3 | Name: ____________________

10 pts, Take-home | Due 9:00pm, Feb 3, 2023

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1. Let $G = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 4 & 2 & 2 & -22 \\ 1 & 4 & -10 & -1 \end{bmatrix}$. and $\mathbf{p} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -1 \end{bmatrix}$.

a (2 pts). Calculate the product $G\mathbf{p}$ or explain why it does not exist.

b (6 pts). Express the solution(s) to $G\mathbf{x} = \mathbf{0}$ in parametric form.

c (2 pts). How many solutions are there to $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$? Explain.

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*Solution:*

1b.(Source: 1.4.11-12,1.5.6-12) To solve the homogeneous system, row reduce $G$. There's no need to augment the zero column $\mathbf{0}$ if we remember that it's unchanged by row operations.

| **row operation** | **result** |
|---|---|
| initial matrix | $\begin{matrix} 1 & 1 & -1 & -5 \\ 4 & 2 & 2 & -22 \\ 1 & 4 & -10 & -1 \end{matrix}$ |
| $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 4\mathbf{r}_1$<br>$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$ | $\begin{matrix} 1 & 1 & -1 & -5 \\ 0 & -2 & 6 & -2 \\ 0 & 3 & -9 & 4 \end{matrix}$ |
| $\mathbf{r}_2 \leftarrow -\frac{1}{2}\mathbf{r}_2$ | $\begin{matrix} 1 & 1 & -1 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 3 & -9 & 4 \end{matrix}$ |
| $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\,\mathbf{r}_2$ | $\begin{matrix} 1 & 1 & -1 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$ |
| $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_3$<br>$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + 5\mathbf{r}_3$ | $\begin{matrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$ |
| $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_2$ | $\begin{matrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$ |

Solution in ... parametric form: parametric vector form:

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= 3x_3 \\ x_3 &= \text{ free} \\ x_4 &= 0 \end{aligned} \qquad \mathbf{x} = s\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

1a.(Source: 1.4.5)

$$G\mathbf{p} = \begin{bmatrix} 1 & 1 & -1 & -5 \\ 4 & 2 & 2 & -22 \\ 1 & 4 & -10 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -3 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \\ -10 \end{bmatrix} - \begin{bmatrix} -5 \\ -22 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix} + \mathbf{0} + \begin{bmatrix} 3 \\ -6 \\ 30 \end{bmatrix} + \begin{bmatrix} 5 \\ 22 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$$

1c.(Source: 1.2.19, 1.5.19-20) The vector $\mathbf{p}$ is a solution to $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$. By Theorem 6, p. 49, the solutions to $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$ are all vectors of the form $\begin{bmatrix} 2 \\ 0 \\ -3 \\ -1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$. Therefore, there are infinitely many solutions to $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$.

1c. Alternate solution. $\mathbf{p}$ is a solution to $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$, so the system has at least one solution. Since $G$ contains a free variable, $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$ must have infinitely many solutions.

**(done)**

*Comment:* 1c. A free variable in a *consistent* system implies that the system has infinitely many variables, but the presence of a free variable does not by itself imply that a nonhomogeneous system has infinitely many variables. In 1c., it's important to note that part a. shows that $G\mathbf{x} = \begin{bmatrix} 10 \\ 24 \\ 33 \end{bmatrix}$ has a solution.