No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

Solve or **find the solution** always means to find the general solution, if it exists.

1(31 pts)

a. Write the solution to $P\mathbf{x} = \mathbf{b}$ in parametric vector form, if

$$P = \begin{bmatrix} 1 & 0 & -2 & 0 \\ -2 & -1 & 4 & 1 \\ 1 & -4 & -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

b. Write the solution to $P\mathbf{x} = \mathbf{0}$ in parametric vector form.

c. What is the dimension of $\operatorname{Col} P$?

d. What is the dimension of $(\operatorname{Row} P)^{\perp}$?

2(22 pts). Compute the inverse of $B = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 4 & 4 \end{bmatrix}$ or show that it does not exist.

3(23 pts). Let $E = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix}$.

a. Find all eigenvalues of E.

b. Is E diagonalizable? Why or why not?

4(28 pts).

a. Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 9 \\ -6 \\ 12 \end{bmatrix}$ onto $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$.

b. What is the distance from $\mathbf{x}$ to W?

c. Find the coordinates of $\mathbf{x}$ relative to the basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \right\}$.

d. Find the coordinates of the polynomial $9 - 6t + 12t^2$ relative to the basis for $\mathbb{P}_2$

$$\{1 + 2t - 2t^2,\ 2 + t + 2t^2,\ 2 - 2t - t^2\}$$

5(15 pts). Find the values of h, if any, that will cause the vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix} \right\}$$

to be linearly dependent.

6(10 pts). The transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $\mathbf{x} \mapsto \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$ consists of a rotation about the origin by an angle θ followed by a scaling by a factor r. Find θ and r. Is the rotation in the clockwise or counterclockwise direction?

7(21 pts). Mark each statement True or False. Supporting work not required for full credit. Matrices are denoted in uppercase (e.g., A), vectors in lowercase **bold** (e.g. $\mathbf{b}$), and scalars in lowercase (e.g. c).

a. If $A^T B^T = B^T A^T$ then $AB = BA$.

b. If A is invertible, then $\det(A) = \det(A^{-1})$.

c. The set $\left\{ \begin{bmatrix} a+3b \\ b-2a \\ 3a-4b \end{bmatrix} \middle| \, a, b \text{ real numbers} \right\}$ is a subspace of $\mathbb{R}^3$.

d. If Z is a 3×2 matrix, then one of the equations

$$Z\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Z\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad Z\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

must be inconsistent.

e. If $\mathcal{B}$ is a basis for the vector space V, and if $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}$, then $\mathbf{u}$ must equal $\mathbf{v}$.

f. If the columns of the $n \times n$ matrix K span $\mathbb{R}^n$, then $\mathbf{u}$ must equal $\mathbf{v}$ whenever $K\mathbf{u} = K\mathbf{v}$.

g. If Y is a square matrix with orthogonal columns, then $Y^T Y = I$.

1a(17 pts).(Source: 1.4.11-14) Augment P with $\mathbf{b}$ and row-reduce.

row operation	result
(beginning matrix)	$\begin{matrix} 1 & 0 & -2 & 0 & -1 \\ -2 & -1 & 4 & 1 & 2 \\ 1 & -4 & -2 & 4 & -2 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + 2\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$	$\begin{matrix} 1 & 0 & -2 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 4 & -1 \end{matrix}$
$\mathbf{r}_2 \leftarrow -\mathbf{r}_2$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 4\,\mathbf{r}_2$	$\begin{matrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{matrix}$

$P\mathbf{x} = \mathbf{b}$ has no solution since there's a pivot in the augmented column.

1b(8 pts).(Source: 1.5.5-6) The reduced row echelon form of P is $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the free variables are x_3 and x_4, and the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

1c(3 pts).(Source: 4.5.11-16) $\dim \operatorname{Col} P$ equals the number of pivot columns, or 2.

1d(3 pts).(Source: 4.5.11-16, 6.1.27-28) $(\operatorname{Row} P)^\perp = \operatorname{Nul} P$. Its dimension equals the number of free variables, or 2.

2(22 pts).(Source: 2.2.41-42) Solution one: augment B with the identity and row-reduce.

row operation	result
(none)	$\begin{matrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\,\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1$	$\begin{matrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & -1 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow -\,\mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 3\,\mathbf{r}_3$	$\begin{matrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & -1 & -1 & 1 \\ 0 & 0 & -2 & -2 & -1 & 1 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} \end{matrix}$

Consequently, $B^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

2. Solution two: the matrix of cofactors of $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 4 & 4 \end{bmatrix}$ is

$$\begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -8 & -2 & 4 \\ 0 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix}$$

Find the determinant of B by cofactor expansion along, say, the first row: $\det(B) = 0(-8) + 2(-2) + 2(4) = 4$. Then Cramer's rule tells use that $B^{-1} = \frac{1}{\det(B)} \operatorname{adj}(B)$:

$$B^{-1} = \frac{1}{4} \begin{bmatrix} -8 & -2 & 4 \\ 0 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix}^T = \frac{1}{4} \begin{bmatrix} -8 & 0 & 4 \\ -2 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

3a(19 pts).(Source: 5.2.9-14) The characteristic polynomial for E is $\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{vmatrix}$. Expand along the second row to obtain

$$(4-\lambda) \begin{vmatrix} -\lambda & 2 \\ 3 & 5-\lambda \end{vmatrix} = (4-\lambda)(\lambda^2 - 5\lambda - 6) = (4-\lambda)(\lambda-6)(\lambda+1).$$

The eigenvalues of E are $\lambda = -1$, 4, and 6.

3b(4 pts).(Source: 5.3.11) Since this 3×3 matrix has 3 distinct eigenvalues, each must have geometric multiplicity 1. Therefore, $\mathbb{R}^3$ has a basis consisting of eigenvectors of E, which implies that E is diagonalizable.

4a(11 pts).(Source: 6.3.3-6) Since the given spanning set $\{\mathbf{u}, \mathbf{v}\}$ for W is orthogonal, we can calculate

$$\operatorname{proj}_W \mathbf{x} = \frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} + \frac{\mathbf{v} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{-27}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \frac{36}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 14 \end{bmatrix}.$$

4b(8 pts).(Source: 6.3.15-16) $\|\mathbf{x} - \operatorname{proj}_W \mathbf{x}\| = \left\| \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix} \right\| = \sqrt{16+16+4} = 6.$

4c(6 pts).(Source: 6.2.9-10) The first two coordinates were calculated in part a. Letting $\mathbf{w} = \begin{bmatrix} 2 & -2 & -1 \end{bmatrix}^T$, the third coordinate is $\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} = \frac{18}{9} = 2$. Coordinates are $(-3, 4, 2)$.

4d(3 pts).(Source: 4.4.13-14) Since $\begin{bmatrix} 9 \\ -6 \\ 12 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$, by comparing coefficients of the polynomials we see

$$9 - 6t + 12t^2 = -3(1 + 2t - 2t^2) + 4(2 + t + 2t^2) + 2(2 - 2t - t^2).$$

That is, the coordinates are again $(-3, 4, 2)$.

5(15 pts).(Source: 1.7.11-14,15,18, 3.2.21-22) Calculate the 3×3 determinant below by expansion along its third column:

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & h \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} + h \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2(-2) + h(0) = -4$$

There is no h that would cause the determinant to equal zero, so there is **no** h for which the vectors in question are linearly dependent.

6(10 pts).(Source: 5.5.10-11, 1.9.3-4,7) Solution one: recall that the standard matrix for rotation in the positive (counterclockwise) direction by an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(see p. 101 of our text). Since $\cos^2\theta + \sin^2\theta = 1$, and the norm of the first column of our matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

is $\sqrt{2}$, factor out $\sqrt{2}$:

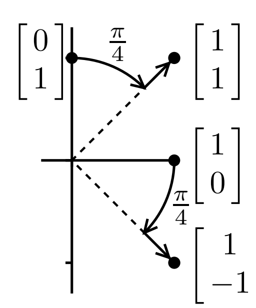
$$A = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and recognize $\pm\frac{1}{\sqrt{2}}$ as the cosine and sine of $-\frac{\pi}{4}$:

$$A = \sqrt{2} \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix}$$

Therefore A rotates vectors $\theta = -\frac{\pi}{4}$ radians in the positive (counterclockwise) direction (that is, $\frac{\pi}{4}$ radians in the clockwise direction) and multiplies the result by $r = \sqrt{2}$.

6. Solution two: multiplication by A sends the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Observe that this rotates both $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $\theta = \frac{\pi}{4}$ radians in the clockwise direction.



Since A sends the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ of norm 1 to the vectors $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ of norm $\sqrt{2}$, the scaling factor must be $\sqrt{2}$.

7a(3 pts).(Source: 2.1.22) **T**. $BA = AB$ because their transposes are equal.: $(BA)^T = A^T B^T = B^T A^T = (AB)^T$.

7b(3 pts).(Source: 3.2.34) **F**. $\det(A^{-1}) = \det(A)^{-1}$.

7c(3 pts).(Source: 4.1.15-18, 4.2.11-14) **T**. Set in question is the span of $\left\{ \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix} \right\}$

7d(3 pts).(Source: 1.4.21+) **T**. If the span of the columns of Z included $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ then it would include all of $\mathbb{R}^3$, since $\mathbb{R}^3$ is spanned by these three vectors. But it's not possible for the 2 columns of Z to span the 3-dimensional $\mathbb{R}^3$.

7e(3 pts).(Source: 4.4.27) **T**. The coordinates of a vector tell us how to write the vecc-tor as a linear combintation of the basis elements. Same coordinates means same linear combintation, hence same vector.

7f(3 pts).(Source: 2.3.20) **T**. See the Invertible Matrix theorem, p. 145 of our text.
If the square matrix K has a pivot in every row, then it has a pivot in every column, and so its columns are linearly independent. Then $K(\mathbf{u} - \mathbf{v}) = \mathbf{0}$ implies $\mathbf{u} - \mathbf{v} = \mathbf{0}$.

7g(3 pts).(Source: 6.2.35) **F**. If Y is a square matrix with orthogonal columns, then Y^TY is a diagonal matrix, not necessarily the identity. For instance, if the columns of Y were the three orthogonal vectors in problem 4c, then $Y^TY = \begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$. In order for $Y^TY = I$, the columns of Y must be orthonormal.