No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Solve or **find the solution** always means to find the general solution, if it exists.

1(16 pts). The set $\mathcal{D} = \{1+t,\ 2+t\}$ is a basis for $\mathbb{P}_1$, the vector space of all polynomials of degree 1 or less.

a. Find the polynomial $\mathbf{p}(t)$ with the coordinates $[\mathbf{p}(t)]_{\mathcal{D}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

b. Find the coordinates $[\mathbf{q}(t)]_{\mathcal{D}}$ of the polynomial $\mathbf{q}(t) = t - 3$.

2a(4 pts). Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$?

2b(8 pts). Find all eigenvalues of A.

2c(10 pts). Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$, or explain why none exist.

3a(10 pts). Find all eigenvalues of $B = \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix}$.

3b(15 pts). Find an invertible matrix Q and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ so that $B = QCQ^{-1}$, or explain why none exist.

4. Define the linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_1$ by $T(\mathbf{p}(t)) = \mathbf{p}(-1) + \mathbf{p}(1)t$.

a(4 pts). Compute $T(2-3t)$. Is $2-3t$ an eigenvector for T?

b(12 pts). Find the matrix of T relative to the standard basis $\{1, t\}$ for $\mathbb{P}_1$.

5(6 pts). Suppose the 5×7 matrix E has rank 3. Find the following.

a. $\dim \operatorname{Col} E$ b. $\dim \operatorname{Col} E^T$ c. $\dim \operatorname{Nul} E$ d. $\dim \operatorname{Nul} E^T$

6(15 pts). Answer **one** of the following parts. Clearly indicate which part you're answering.

a. Show that if λ is an eigenvalue of an invertible matrix G, then λ^{-1} must be an eigenvalue of G^{-1}. Begin your solution with a definition of what it means for λ to be an eigenvalue of G.

b. Show that if the matrices U and W are similar, then $U+I$ and $W+I$ are also similar. Begin your solution with a definition of what it means for U and W to be similar.

1a(4 pts).(Source: 4.4.1-4) $\mathbf{p}(t) = -2(1+t) + 2(2+t)$. If you wish, you could simplify $\mathbf{p}(t)$ to $-2 - 2t + 4 + 2t = 2$, that is, the constant function 2.

1b(12? pts).(Source: 4.4.13-14) The coordinates of $t-3$ are the numbers x and y for which

$$x(1+t) + y(2+t) = t - 3 \tag{0.1}$$

for all t. Evaluate at t-values to obtain a system of equations for x and y.

$$\begin{aligned} t &= -2 \\ t &= -1 \end{aligned} \implies \begin{aligned} -x + 0y &= -5 \\ 0x + 1y &= -4 \end{aligned} \implies \begin{aligned} x &= 5 \\ y &= -4 \end{aligned}$$

That is, $[\mathbf{q}(t)]_{\mathcal{D}} = [5 \quad 4]^T$.
You could also have found x and y by solving the system that results from equating the coefficients of 1 and t in (0.1):

$$\begin{aligned} &\text{1-coefficient} \\ &t\text{-coefficient} \end{aligned} \implies \begin{aligned} x + 2y &= -3 \\ x + y &= 1 \end{aligned}$$

2a(4 pts).(Source: 5.1.3) $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

2b(8 pts).(Source: 5.2.1) The eigenvalues of A are the zeros of its characteristic polynomial:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) - (-4) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

so $\lambda = 3$ is the only eigenvalue.

2c(10 pts).(Source: 5.3.9) The existence of such P and D means that A is diagonalizable. For that to occur, the eigenspace of A associated to $\lambda = 3$ must have dimension 2. But

$$A - 3I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

has only one non-pivot column. The eigenspace has dimension 1, so A is not diagonalizable.

3a(10 pts).(Source: 5.5.1-5) The characteristic polynomial of B is

$$|B - \lambda I| = \begin{vmatrix} 5-\lambda & 6 \\ -3 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) - (-18) = \lambda^2 - 4\lambda + 13$$

To find the zeros, either use the quadratic formula or complete the square, as shown here:

$$\lambda^2 - 4\lambda + 4 = -13 + 4 \implies (\lambda - 2)^2 = -9 \implies \lambda - 2 = \pm 3i \implies \lambda = 2 \pm 3i$$

3b(15 pts).(Source: 5.3.9) Q and C exist because B has non-real eigenvalues. If we use $\lambda = 2 - 3i$, the resulting C is the rotation-and-scaling matrix

$$C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}.$$

To find Q, we must find a basis for the eigenspace of B corresponding to this eigenvalue.

$$B - (2-3i)I = \begin{bmatrix} 5-(2-3i) & 6 \\ -3 & -1-(2-3i) \end{bmatrix} = \begin{bmatrix} 3+3i & 6 \\ -3 & -3+3i \end{bmatrix}$$

Since this matrix is singular, the first and second rows are linearly dependent, and so the the matrix is row equivalent to

$$\begin{bmatrix} -3 & -3+3i \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

The vector $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is a basis for the eigenspace. Use its real and imaginary parts for the columns of Q:

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

(done)

If you used the eigenvalue $\lambda = 2 + 3i$, the results would be

$$C = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

Both answers are correct, since

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix}$$

4.(Source: 5.4.3-4,15-16)
4a(4 pts). If $\mathbf{p}(t) = 2 - 3t$, then $\mathbf{p}(-1) = 5$ and $\mathbf{p}(1) = -1$, and so $T(2-3t) = 5 - t$. Since this is not a scalar multiple of $2 - 3t$, $2 - 3t$ is not an eigenvector for T.

4b(12 pts). Letting $\mathcal{B}$ stand for the standard basis $\{1, t\}$, the matrix M must satisfy this diagram:

$$\begin{array}{ccc} \mathbb{P}_1 & \xrightarrow{T} & \mathbb{P}_1 \\ {[\ ]_\mathcal{B}}\downarrow & & \downarrow{[\ ]_\mathcal{B}} \\ \mathbb{R}^2 & \xrightarrow{M} & \mathbb{R}^2 \end{array}$$

That is, $M[\mathbf{p}]_\mathcal{B} = [T(\mathbf{p})]_\mathcal{B}$ for each $\mathbf{p}$ in $\mathbb{P}_1$. Evaluate T at the basis elements:

$$T(1) = 1 + 1t$$
$$T(t) = -1 + 1t$$

Therefore

$$\text{1st column of } M = M\mathbf{e}_1 = [T(1)]_\mathcal{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\text{2st column of } M = M\mathbf{e}_2 = [T(2)]_\mathcal{B} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and so

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

5(6 pts).(Source: 4.5.33-38) rank $E = \dim \operatorname{Col} E = \dim \operatorname{Row} E = \dim \operatorname{Col} E^T$, so a. = b.= 3. Since the rank of a matrix equals its number of pivot columns, and the dim Nul of a matrix equals its number of non-pivot columns, dim Nul equals the number of columns minus rank. Therefore, c.= the number of columns of E minus rank E, or $7-3=4$, and d.= the number of columns of E^T minus rank E^T, or $5-3=2$.

6a(15 pts).(Source: 5.1.33) λ is an eigenvalue of G if $G\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$. For such an $\mathbf{x}$,

$$G\mathbf{x} = \lambda\mathbf{x} \implies G^{-1}G\mathbf{x} = G^{-1}\lambda\mathbf{x} \implies I\mathbf{x} = \lambda G^{-1}\mathbf{x}$$

Since a matrix is invertible iff non of its eigenvalues is 0, we can divide both sides by λ to obtain

$$\lambda^{-1} \implies \mathbf{x} = G^{-1}\mathbf{x},$$

proving that λ^{-1} is an eigenvalue of G^{-1}.

6b(15 pts).(Source: 5.2.32) For two matrices U and W to be similar means that there's an invertible matrix P for which

$$U = PWP^{-1}$$

But then

$$U + I = PWP^{-1} + PIP^{-1} = P(W+I)P^{-1},$$

proving that $U + I$ and $W + I$ are similar.