No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Solve or **find the solution** always means to find the general solution, if it exists.

1(10 pts). Calculate the product, if it exists:

a. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$

2(25 pts). Let $V = \left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix}$.

a. Express $\mathbf{b}$ as a linear combination of the elements of V or explain why this is not possible.

b. Is every vector in $\mathbb{R}^3$ in the span of V? Why or why not?

3(18 pts). Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the rule

$$T(\mathbf{x}) = \begin{bmatrix} 5 & 11 & -11 \\ 1 & 2 & -3 \\ -1 & -1 & 7 \end{bmatrix} \mathbf{x}.$$

Determine whether $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ for some $\mathbf{x}$, and, if so, if $\mathbf{x}$ is unique.

4(35 pts). Let $D = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & -2 & 0 & 2 \\ 1 & 2 & 2 & -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -2 \\ -6 \\ 7 \end{bmatrix}$.

a. Find the parametric vector form of the general solution to $D\mathbf{x} = \mathbf{u}$.

b. State the general solution to $D\mathbf{x} = \mathbf{0}$.

c. Are the columns of D linearly independent? Why or why not?

5(12 pts). Answer **one** of the following parts. Clearly indicate which part you're answering.

a. Suppose that there's a vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has exactly one solution. Explain why $A\mathbf{x} = \mathbf{0}$ has exactly one solution.

b. Suppose C is an $m \times n$ matrix with $m > n$. Explain why there must be a vecor $\mathbf{d}$ in $\mathbb{R}^m$ for which $C\mathbf{x} = \mathbf{d}$ has no solutions.

1a(4 pts).(Source: 1.4.1,3) Does not exist. The matrix-vector product $A\mathbf{x}$ requires the number of columns of A equal the number of rows of $\mathbf{x}$.

1b(6 pts).

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$$

2a(21 pts).(Source: 1.3.11-12) To solve the linear system, augment and row reduce.

row operation	result
initial matrix	$\begin{matrix} 0 & 2 & -1 & 8 \\ 1 & 0 & -2 & 6 \\ -2 & 6 & 2 & 10 \end{matrix}$
$\mathbf{r}_1 \rightleftarrows \mathbf{r}_2$	$\begin{matrix} 1 & 0 & -2 & 6 \\ 0 & 2 & -1 & 8 \\ -2 & 6 & 2 & 10 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1$	$\begin{matrix} 1 & 0 & -2 & 6 \\ 0 & 2 & -1 & 8 \\ 0 & 6 & -2 & 22 \end{matrix}$

row operation	result
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_2$	$\begin{matrix} 1 & 0 & -2 & 6 \\ 0 & 2 & -1 & 8 \\ 0 & 0 & 1 & -2 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + \mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 + 2\mathbf{r}_3$	$\begin{matrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -2 \end{matrix}$
$\mathbf{r}_2 \leftarrow \frac{1}{2}\mathbf{r}_2$	$\begin{matrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{matrix}$

Conclusion: $\mathbf{b}$ is (uniquely) expressible as a linear combination of the vectors in V:

$$\begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

2b(4 pts).(Source: 1.4.18-22) Yes, V spans $\mathbb{R}^3$ because the matrix

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -2 \\ -2 & 6 & 2 \end{bmatrix}$$

has a pivot in every row (of its row echelon form).

3(18 pts).(Source: 1.8.3-7) To solve the linear system, augment and row reduce

row operation	**result**
initial matrix	$\begin{matrix} 5 & 11 & -11 & 1 \\ 1 & 2 & -3 & 0 \\ -1 & -1 & 7 & 3 \end{matrix}$
$\mathbf{r}_1 \rightleftarrows \mathbf{r}_2$	$\begin{matrix} 1 & 2 & -3 & 0 \\ 5 & 11 & -11 & 1 \\ -1 & -1 & 7 & 3 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 5\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1$	$\begin{matrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 3 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_2$	$\begin{matrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{matrix}$

This system is inconsistent, since there's a pivot in the last column of the augmented matrix. There's no $\mathbf{x}$ satisfying $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. (That is, $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ isn't in the range of T.)

4a(25 pts).(Source: 1.5,19,20) Augment and row reduce to solve.

row operation	**result**
initial matrix	$\begin{matrix} 1 & -1 & 2 & 2 & -2 \\ 0 & -2 & 0 & 2 & -6 \\ 1 & 2 & 2 & -1 & 7 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$	$\begin{matrix} 1 & -1 & 2 & 2 & -2 \\ 0 & -2 & 0 & 2 & -6 \\ 0 & 3 & 0 & -3 & 9 \end{matrix}$
$\mathbf{r}_2 \leftarrow -\frac{1}{2}\mathbf{r}_2$	$\begin{matrix} 1 & -1 & 2 & 2 & -2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 3 & 0 & -3 & 9 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_2$	$\begin{matrix} 1 & -1 & 2 & 2 & -2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$

Solution in . . . parametric form: parametric vector form (required):

$$\begin{aligned} x_1 &= 1 - 2x_3 - x_4 \\ x_2 &= 3 + x_4 \\ x_3 &= \text{ free} \\ x_4 &= \text{ free} \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

4b(5 pts).(Source: 1.5.5,6) The solution to the homogeneous system in . . .
parametric form: parametric vector form:

$$\begin{aligned} x_1 &= -2x_3 - x_4 \\ x_2 &= x_4 \\ x_3 &= \text{ free} \\ x_4 &= \text{ free} \end{aligned}$$

$$\mathbf{x} = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Either form was acceptable in 4b.

4c(5 pts).(Source: 1.7.5-8) The columns of D are not linearly independent since D doesn't have a pivot in every column.

5(12 pts).

a.(Source: 1.5.38) Assume that $A\mathbf{x} = \mathbf{b}$ has exactly one solution for some vector $\mathbf{b}$. It's impossible that A contains any free variables, since this would cause the consistent system $A\mathbf{x} = \mathbf{b}$ to have infinitely many solutions.
A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has the trivial solution $\mathbf{x} = \mathbf{0}$ and has infinitely many solutions iff A has free variables. Since A has none, $A\mathbf{x} = \mathbf{0}$ must have exactly one solution.

a. (Alternate solution) The homogeneous system $A\mathbf{x} = \mathbf{0}$ always has the solution $\mathbf{x} = \mathbf{0}$ Suppose $\mathbf{p}$ is the unique vector solving $A\mathbf{x} = \mathbf{b}$ and that $\mathbf{u}$ is any solution to $A\mathbf{x} = \mathbf{0}$. Then $\mathbf{p} + \mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$, since $A(\mathbf{p} + \mathbf{u}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Since this system has only one solution, $\mathbf{p} + \mathbf{u} = \mathbf{p}$. Subtract $\mathbf{p}$ from both sides to obtain $\mathbf{u} = \mathbf{0}$. Consequently, every solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$, and so this homogeneous system has exactly one solution.

b.(Source: 1.4.41) Assume that C is an $m \times n$ matrix with $m > n$. Any matrix has at most one pivot per column, so C has at most n pivots. Since this is less than the number of its rows, C cannot have a pivot in each row, and therefore cannot span $\mathbb{R}^m$. That is, there's at least one $\mathbf{d} \in \mathbb{R}^m$ for which $C\mathbf{x} = \mathbf{d}$ has no solution.