Due by end-of-class, 10/17/2014, either in class, in my mailbox (339 RS Small), or in my emailbox (kunklet@cofc.edu).

Use any outside materials that you want (other than a tutor), work with your classmates or not, but do **not** simply submit a solution that you got from someone else. Your proofs should demonstrate not just the correctness of the statements below but also your complete understanding of their meaning. Pay particular attention to instructions I gave you in "How to write subspace proofs," available at http://kunklet.people.cofc.edu/syll203_2015-10.html. I will penalize you for sloppy, ambiguous, ungrammatical, or poor wording, including no wording.

Write your solution on whatever paper you prefer, using as many pages as necessary. Please be neat. You needn't turn in this sheet.

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1 (10 pts). Define $L = \{f \in C[0, 2\pi] \mid \int_0^{2\pi} f(x) \sin x \, dx = 0\}$.

For instance, the function $f(x) = \cos x$ is in L because

$$\int_0^{2\pi} \cos x \sin x \, dx = \frac{1}{2} \sin^2 x \Big|_0^{2\pi} = \frac{1}{2}(\sin^2(2\pi) - \sin^2 0) = 0.$$

Prove that L is a subspace of $C[0, 2\pi]$.

Solutions

1. In the vector space $C[0, 2\pi]$, the zero vector $\mathbf{0}$ is the constant function whose output is the number 0 for every input. That is, $\mathbf{0}(x) = 0$ for all $x \in [0, 2\pi]$. $\mathbf{0}$ is in H because $\int_0^{2\pi} \mathbf{0}(x) \sin x \, dx = \int_0^{2\pi} 0 \sin x \, dx = \int_0^{2\pi} 0 \, dx = 0$.

Suppose that u and v are in H, so that $\int_0^{2\pi} u(x) \sin x \, dx = \int_0^{2\pi} v(x) \sin x \, dx = 0$. Then $u + v$, the function defined by the rule $(u + v)(x) = u(x) + v(x)$, is in H because

$$\begin{aligned} &\int_0^{2\pi} \big(u + v\big)(x) \sin x \, dx \\ &= \int_0^{2\pi} \big(u(x) + v(x)\big) \sin x \, dx \\ &= \int_0^{2\pi} \big(u(x) \sin x + v(x) \sin x\big) \, dx \\ &= \int_0^{2\pi} u(x) \sin x \, dx + \int_0^{2\pi} v(x) \sin x \, dx \\ &= 0 + 0 = 0. \end{aligned}$$

Thus H is closed under vector addition.

Suppose that u is in H, so that $\int_0^{2\pi} u(x) \sin x \, dx = 0$ and that c is a scalar. Then cu, the function defined by the rule $(cu)(x) = c \cdot u(x)$, is in H because

$$\begin{aligned}
&\int_0^{2\pi} (cu)(x) \sin x \, dx = 0 \\
&= \int_0^{2\pi} c \cdot u(x) \sin x \, dx = 0 \\
&= c \cdot \int_0^{2\pi} u(x) \sin x \, dx = 0 \\
&= c \cdot 0 = 0.
\end{aligned}$$

Thus H is closed under scalar multiplication.

Since H is nonempty and closed under vector addition and scalar multiplication, H is a subspace of $C[0, 2\pi]$. QED