1. Define $H = \{f \in C^2(-\infty, \infty) \mid f'' - 2f' + f = 0\}$. That is, $H$ consists of all twice-continuously differentiable functions $f$ on $\mathbb{R}$ for which $f''(x) - 2f'(x) + f(x) = 0$ for all real numbers $x$.

For example, the function $f(x) = xe^x$ is in $H$, since

$$f(x) = xe^x \quad f'(x) = xe^x + e^x \quad f''(x) = xe^x + 2e^x$$

and so

$$f''(x) - 2f'(x) + f(x) = xe^x + 2e^x - 2(xe^x + e^x) + xe^x = xe^x - 2xe^x - 2e^x + xe^x = 0$$

Prove that $H$ is a subspace of $C^2(-\infty, \infty)$.

2. Define $K = \{f \in C^2(-\infty, \infty) \mid f''(x) - 2f'(x) + f(x) = \sin x \text{ for all real } x\}$. Prove that $K$ is not a subspace of $C^2(-\infty, \infty)$.

3. Find a nonzero function in $H$ (other than a scalar multiple of $xe^x$) and another in $K$. Hint: try exponentials and trig functions. “Nonzero function” means a function other than $0$.

Solutions

1. In the vector space $C^2(-\infty, \infty)$, the zero vector $0$ is the constant function whose output is the number $0$ for every input: $x \in \mathbb{R} \rightarrow 0(x) = 0$. Because the derivative of a constant is $0$, both $0'$ and $0''$ equal $0$. Therefore $0'' - 20' + 0 = 0 + 0 + 0 = 0$, so $0$ is in $H$.

Suppose that $u$ and $v$ are in $H$, so that both $u'' - 2u' + u = 0$ and $v'' - 2v' + v = 0$. Then $u + v$, the function defined by the rule $(u + v)(x) = u(x) + v(x)$, is in $H$ because

$$(u + v)' = u' + v' \quad (u + v)'' = u'' + v'' - 2u' - 2v' + u + v$$

$$(u + v)'' - 2(u + v)' + (u + v)$$

$$(u'' + v'') - 2(u' + v') + u + v$$

$$(u'' - 2u' + u + v'' - 2v' + v)$$

$= 0 + 0 = 0.$$
Thus $H$ is closed under vector addition.

Suppose that $u$ is in $H$, so that $u'' - 2u' + u = 0$ and that $c$ is a scalar. Then $cu$, the function defined by the rule $(cu)(x) = c \cdot u(x)$, is in $H$ because $(cu)' = cu'$ and $(cu)'' = cu''$, and therefore

$$
(cu)'' - 2(cu)' + (cu) = cu'' - 2cu' + cu = c(u'' - 2u' + u) = c \cdot 0 = 0.
$$

Thus $H$ is closed under scalar multiplication.

Since $H$ is nonempty and closed under vector addition and scalar multiplication, $H$ is a subspace of $C^2(-\infty, \infty)$. QED

2. As noted above, $0'' - 20' + 0 = 0$, not $\sin x$, so $0$ isn’t in $K$. Therefore $K$ is not a vector space. QED

3. The function $e^x$ is in $H$ because $(e^x)'' - 2(e^x)' + e^x = e^x - 2e^x + e^x = 0$ for all real numbers $x$. (In fact, $H$ is the span of the functions $e^x$ and $xe^x$.)

The function $\frac{1}{2} \cos x$ is in $K$, since

$$(\frac{1}{2} \cos x)' = -\frac{1}{2} \sin x$$

and

$$(\frac{1}{2} \cos x)'' = (-\frac{1}{2} \sin x)' = -\frac{1}{2} \cos x$$

and therefore for all real numbers $x$,

$$
(\frac{1}{2} \cos x)' - 2(\frac{1}{2} \cos x)' + \frac{1}{2} \cos x = -\frac{1}{2} \cos x - 2(-\frac{1}{2} \sin x) + \frac{1}{2} \cos x = \sin x.
$$

(In fact, $K$ equals $\left\{ \frac{1}{2} \cos x + c_1 e^x + c_2 xe^x \mid c_1, c_2 \in \mathbb{R} \right\}$.) QED