MATH 203–01 (Kunkle), Exam 3 Name: ____________________
100 pts, 50 minutes Nov. 14, 2014

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

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1. Let $A = \begin{bmatrix} 6 & 1 & -2 \\ 0 & 2 & 0 \\ 6 & 2 & -1 \end{bmatrix}$.

a (10 pts). Find all eigenvalues of A.

b (10 pts). Determine whether A is diagonalizable.

2 (3 pts). Determine whether $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 6 & -1 \\ 4 & 2 \end{bmatrix}$.

3. Let $E = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ and let $\mathcal{B}$ be the basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

a (8 pts). Find the $\mathcal{B}$-matrix for E. Hint: compute the product $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

b (14 pts). Find a basis $\mathcal{C}$ of $\mathbb{R}^2$ for which the $\mathcal{C}$-matrix of E is diagonal.

4. Let $\mathbf{x} = [\,3 \quad -1 \quad 2\,]^T$.

a (4 pts). Find a unit vector in the direction of $\mathbf{x}$.

b (5 pts). Find the orthogonal projection of $\mathbf{x}$ onto the span of the vector $[\,-1 \quad 4 \quad -1\,]^T$.

c (10 pts). The set $\mathcal{D} = \left\{ \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express $\mathbf{x}$ as a linear combination of the elements of of $\mathcal{D}$.

5. Let $F = \begin{bmatrix} -5 & 20 \\ -2 & 7 \end{bmatrix}$.

a (10 pts). Find the eigenvalues of F.

b (16 pts). Find matrices P and C so that the transformation $\mathbf{x} \mapsto C\mathbf{x}$ is a composition of a rotation and a scaling and $F = PCP^{-1}$. (You are not required to find P^{-1}.)

6 (10 pts). Answer **either** a. **or** b. Clearly indicate which part you're answering.

a. Prove the following. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in $\mathbb{R}^4$, if $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$, and if $\mathbf{x}$ is in span$\{\mathbf{v}, \mathbf{w}\}$, then $\mathbf{u} \perp \mathbf{x}$. Include in your proof a definition of the symbols "$\mathbf{u} \perp \mathbf{v}$" and "span$\{\mathbf{v}, \mathbf{w}\}$."

b. Prove that if U is an $m \times n$ matrix with orthonormal columns, and if V is an $n \times p$ matrix with orthonormal columns, then the columns of UV must also be orthonormal.

1a. $\begin{vmatrix} 6-\lambda & 1 & -2 \\ 0 & 2-\lambda & 0 \\ 6 & 2 & -1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ 6 & -1-\lambda \end{vmatrix} = (2-\lambda)\big((6-\lambda)(-1-\lambda)-(-12)\big)$

$= (2-\lambda)(\lambda^2-5\lambda+6) = (2-\lambda)(\lambda-2)(\lambda-3).$ $\lambda = \underline{2 \text{ or } 3}$.

b. Check eigenvalue of algebraic multiplicity 2:

$\lambda=2$;
$A-2I = \begin{pmatrix} 4 & 1 & -2 \\ 0 & 0 & 0 \\ 6 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 4 & 1 & -2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. One free var mean $\dim(A-2I)=1$, not 2. A is not diagonalizable.

Alt.

a. $\begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ -6 & 2 & 6-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{vmatrix} = (3-\lambda)\big((-1-\lambda)(6-\lambda)-(-12)\big)$

$= (3-\lambda)(\lambda^2-5\lambda+6) = (3-\lambda)(\lambda-2)(\lambda-3).$ $\lambda = 2$ or 3.

b. $\lambda=3$. $A-3I = \begin{pmatrix} -4 & 1 & 2 \\ 0 & 0 & 0 \\ -6 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} -4 & 1 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. one free var. $\dim(A-3I)=1 \neq 2$. A not diagonalizable.

2. $\begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. yes. Alt: $\begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix} \neq \lambda\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. No.

3a. $P = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Observe from hint $P\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = I$, so $P^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

B-matrix $= P^{-1}EP = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -9 & -6 \end{pmatrix}$.

Alt: $P^{-1}EP = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ -16 & -6 \end{pmatrix}$.

3b. $|E-\lambda I| = \begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = \lambda^2-9 = 0 \Rightarrow \lambda = 3, -3$.

$\lambda=3$: $E-3I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ $x = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda=-3$: $E+3I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $x = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

alt. $\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 2$.

$\lambda=2 \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ @ $\lambda=-2$, $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. $\mathcal{B} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

4a. $\frac{1}{\|x\|}x = \frac{1}{\sqrt{9+1+4}}\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{14} \\ -1/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}$. Alt: $\begin{pmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}$.

b. $u = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$. $\text{proj}_u x = \frac{x\cdot u}{u \cdot u}u = \frac{-9}{18}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -2 \\ 1/2 \end{pmatrix}$. Alt $\frac{9}{18}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 2 \\ -1/2 \end{pmatrix}$.

4 c. $x = \frac{x\cdot u}{u\cdot u}u + \frac{x\cdot v}{v\cdot v}v + \frac{x\cdot w}{w\cdot w}w$

$= \frac{-1}{2}\begin{pmatrix}-1\\4\\-1\end{pmatrix} + \frac{9}{9}\begin{pmatrix}2\\1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix}-1\\4\\-1\end{pmatrix} + 1\begin{pmatrix}2\\1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$

alt: $\frac{x\cdot u}{u\cdot u}u + \frac{x\cdot v}{v\cdot v}v + \frac{x\cdot w}{w\cdot w}w$

$= \frac{1}{2}\begin{pmatrix}-1\\4\\-1\end{pmatrix} + \frac{9}{9}\begin{pmatrix}2\\1\\2\end{pmatrix} + \frac{-1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}-1\\4\\-1\end{pmatrix} + 1\begin{pmatrix}2\\1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}.$

5a. $\begin{vmatrix}-5-\lambda & 20\\ -2 & 7-\lambda\end{vmatrix} = (-5-\lambda)(7-\lambda) - (-40) = \lambda^2 - 2\lambda + 5 = \lambda^2 - 2\lambda + 1 + 4$

$= (\lambda-1)^2 + 4 = 0 \Rightarrow \lambda - 1 = \pm 2i \Rightarrow \lambda = 1 \pm 2i.$

b. Let $\lambda = 1-2i$. $F - \lambda I = \begin{pmatrix}-5-1+2i & 20\\ -2 & 7-1+2i\end{pmatrix} = \begin{pmatrix}-6+2i & 20\\ -2 & 6+2i\end{pmatrix} \sim \begin{pmatrix}1 & -3-i\\ -3+i & 10\end{pmatrix}$

$\sim^{*} \begin{pmatrix}1 & -3-i\\ 0 & 0\end{pmatrix}$. Let x_2 (= free) $= 1$. Then $x_1 = 3+i$ $\quad x = \begin{pmatrix}3+i\\1\end{pmatrix}$

*b.c. $(-3+i)(1, -3-i) = (-3+i, 10)$

$P = (\operatorname{Re}(x)\ \ \operatorname{Im}(x)) = \begin{pmatrix}3 & 1\\ 1 & 0\end{pmatrix}$. $C = \begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$.

Alt: a. $\begin{vmatrix}-5-\lambda & 15\\ -3 & 7-\lambda\end{vmatrix} = (-5-\lambda)(7-\lambda) + 45 = \lambda^2 - 2\lambda + 10 = \lambda^2 - 2\lambda + 1 + 9$

$= (\lambda-1)^2 + 9 = 0 \Rightarrow \lambda - 1 = \pm 3i \Rightarrow \lambda = 1 \pm 3i.$

b. Let $\lambda = 1-3i$. $F - \lambda I = \begin{pmatrix}-5-1+3i & 15\\ -3 & 7-1+3i\end{pmatrix} = \begin{pmatrix}-6+3i & 15\\ -3 & 6+3i\end{pmatrix} \sim \begin{pmatrix}1 & -2-i\\ -2+i & 5\end{pmatrix}$

$\sim \begin{pmatrix}1 & -2-i\\ 0 & 0\end{pmatrix}$. Let $x_2 = 1$, $x_1 = 2+i$. $x = \begin{pmatrix}2+i\\1\end{pmatrix}$. $P = \begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}$. $C = \begin{pmatrix}1 & -3\\ 3 & 1\end{pmatrix}$.

6a. $u \perp v$ means $u\cdot v = 0$. span$\{v, w\}$ is the collection of all linear combinations of v & w.
Given $u\cdot v = u\cdot w = 0$, $x = av + bw$ for some scalars a & b.
Then $u\cdot x = u\cdot(av + bw) = a(u\cdot v) + b(u\cdot w) = a0 + b0 = 0$,
so $u \perp x$.

6b. Use Fact: A has orthonormal columns iff $A^TA = I$.
Given: $U^TU = I_{(n\times n)}$, $V^TV = I_{(p\times p)}$.
Then $(UV)^T(UV) = V^T\underbrace{U^TU}_{=I}V = V^TV = I$, so
columns of UV are orthonormal.