No notes, books, electronic devices, or outside materials of any kind.

Source

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

6.2 → 1. Let $\mathbf{y} = [4 \quad 3 \quad -3]^T$ and $\mathbf{w} = [1 \quad 0 \quad -2]^T$.

a (8 pts). Write $\mathbf{y}$ as the sum of a vector parallel to $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$.
b (4 pts). Based on your work in part a, what would you say is the distance from $\mathbf{y}$ to the line spanned by $\mathbf{w}$?

6.3 or higher → 2 (20 pts). Apply the Gram-Schmidt process to the three vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix}$.

6.3 or higher → 3. Let $L = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & -3 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -4 \\ 0 \\ 8 \\ 0 \end{bmatrix}$.

a (12 pts). Find the least-squares solution to $L\mathbf{x} = \mathbf{b}$.
b (4 pts). Use your answer to part a to find the projection of $\mathbf{b}$ onto $\operatorname{Col} L$.

4 → 4 (4 pts). The set $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Find the vector in $\mathbb{R}^3$ whose coordinates with respect to this basis are $[0 \quad 3 \quad 1]^T$.

3 → 5 (8 pts). Compute the determinant of $D = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ -1 & -1 & 0 & -1 \\ -3 & 1 & -2 & 0 \end{bmatrix}$

5 → 6a (10 pts). Find the eigenvalues of $H = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$.

4 → 6b (4 pts). Find the rank of H (in part a above).

5 → 7a (8 pts). Determine which of the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \end{bmatrix} \right\}$ are eigenvectors of $M = \begin{bmatrix} 5 & -2 \\ 6 & -3 \end{bmatrix}$.

5 → 7b (4 pts). **Either** find real, square matrices P and D so that D is diagonal and $PDP^{-1} = M$ (from part a above) **or** explain why this is not possible.
(In the first case, you are not required to find P^{-1}.)

1 → 8. Let $A = \begin{bmatrix} 2 & 4 & 0 & -4 & 6 \\ -1 & -1 & -1 & 1 & -2 \\ 0 & -3 & 3 & 4 & -3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -8 \\ 2 \\ 7 \end{bmatrix}$.

a (23 pts). Write the general solution to $A\mathbf{x} = \mathbf{c}$ in parametric vector form.
b (3 pts). Write the general solution to $A\mathbf{x} = \mathbf{0}$ in parametric vector form.
c (2 pts). Do the columns of A span $\mathbb{R}^3$?
d (2 pts). Are the columns of A linearly independent?

e (2 pts). Define $T : \mathbb{R}^5 \to \mathbb{R}^3$ by the rule $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one?

f (2 pts). Is T onto?

2 → 9. Let $G = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 1 & -1 \\ 3 & -3 & 4 \end{bmatrix}$, and let $H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

a (12 pts). Find G^{-1}.

b (8 pts). Find the matrix X for which $XH = G$.

c (8 pts). Find $(GH)^{-1}$.

2, 6.1 → 10 (20 pts). Suppose that

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Find the following or state that it does not exist.

a. $C\mathbf{u}$ b. CB c. BC

d. $\mathbf{u}\mathbf{v}^T$ e. $\mathbf{v}^T\mathbf{w}$ f. $(-101C)B$

g. $B^T C^T$ h. $\|\mathbf{w}\|$ ← 6.1 → i. $\|-101\mathbf{w}\|$

4 → 11 (8 pts). Suppose that P and R are row equivalent, where

$$P = \begin{pmatrix} 3 & q & 1 & r & s \\ 0 & 0 & 2 & t & u \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & p \end{pmatrix}$$

Find a basis for each of the following. (If any of these are the same, say so and you needn't write the basis twice.)

a. $\operatorname{Row} P$ b. $\operatorname{Row} R$ c. $\operatorname{Col} P$ d. $\operatorname{Col} R$

11, continued (2 pts). Find each of the following, or state that more information is needed.

e. $\dim \operatorname{Nul} P$ f. $\dim \operatorname{Nul} R$

4 → 12a (5 pts). Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three vectors in a vector space V. State what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent.

12b (5 pts). State what it means for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to span V. (Don't use the word "span" in your definition.)

12c (6 pts). Are the three polynomials $\{\ 1\ ,\ (t-1)(t+1)\ ,\ t^3+2\ \}$ in $\mathbb{P}_3$ linearly independent? (Recall that $\mathbb{P}_n$ is the space of all polynomials of degree at most n.)
No one-word answers. Supporting work required.

12d (6 pts). Do the three polynomials $\{\ t\ ,\ 2t+1\ ,\ t-1\ \}$ span $\mathbb{P}_1$?
No one-word answers. Supporting work required.

1a. Projection of y onto span{w} $= \frac{y\cdot w}{w\cdot w} w = \frac{10}{5}\begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}2\\0\\-4\end{pmatrix} = \hat{y}$.

$y-\hat{y} = \begin{pmatrix}4\\3\\-3\end{pmatrix} - \begin{pmatrix}2\\0\\-4\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix}$. $y = \hat{y} + (y-\hat{y}) = \underbrace{\begin{pmatrix}2\\0\\-4\end{pmatrix}}_{\parallel w} + \underbrace{\begin{pmatrix}2\\3\\1\end{pmatrix}}_{\perp w}$.

b. dist from y to span{w} $= \|y-\hat{y}\| = \|(2,3,-1)^T\| = \sqrt{4+9+1} = \sqrt{14}$.

2. $u_1 = \begin{pmatrix}2\\-1\\1\end{pmatrix}$. $u_2 = x_2 - \text{proj}_{u_1} x_2 = x_2 - \frac{x_2\cdot u_1}{u_1\cdot u_1}u_1 = \begin{pmatrix}-1\\3\\-1\end{pmatrix} - \frac{-6}{6}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}$.

$u_3 = x_3 - \frac{x_3\cdot u_1}{u_1\cdot u_1}u_1 - \frac{x_3\cdot u_2}{u_2\cdot u_2}u_2 = \begin{pmatrix}-7\\1\\3\end{pmatrix} - \frac{-12}{6}\begin{pmatrix}2\\-1\\1\end{pmatrix} - \frac{-5}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\5\end{pmatrix}$.

(check: $u_2 \perp u_1$ ✓ $u_3 \perp u_1$ ✓ $u_3 \perp u_2$ ✓)

3a. LS solution to $Lx=b$ is solution to $L^TLx = L^Tb$.

$(L^TL \,\vdots\, L^Tb) = \left(\begin{array}{cc|c}4 & 4 & -12\\4 & 14 & -32\end{array}\right) \sim \begin{pmatrix}1 & 1 & 3\\2 & 7 & -16\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 3\\0 & 5 & -10\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 3\\0 & 1 & -2\end{pmatrix}$

$\sim \begin{pmatrix}1 & 0 & 5\\0 & 1 & -2\end{pmatrix}$. $\hat{x} = \begin{pmatrix}5\\-2\end{pmatrix}$.

3b. $\hat{b} = L\hat{x} = \begin{pmatrix}1 & 2\\1 & 0\\-1 & -3\\-1 & 1\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix} = \begin{pmatrix}1\\5\\1\\-7\end{pmatrix}$.

4. If $[x]_B = \begin{pmatrix}0\\3\\1\end{pmatrix}$, then $x = \begin{pmatrix}2 & -1 & 0\\0 & 1 & 0\\0 & 0 & 4\end{pmatrix}\begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}-3\\3\\4\end{pmatrix}$.

5. Expand along column 3. $|D| = -(-2)\begin{vmatrix}1 & -1 & 2\\0 & 2 & 1\\-1 & -1 & -1\end{vmatrix}$. $r_3 \leftarrow r_3 + r_1$

$= 2\begin{vmatrix}1 & -1 & 2\\0 & 2 & 1\\0 & -2 & 1\end{vmatrix} = 2\begin{vmatrix}1 & -1 & 2\\0 & 2 & 1\\0 & 0 & 2\end{vmatrix} = 2\cdot1\cdot2\cdot2 = 8$.

6a. $|H-\lambda I| = \begin{vmatrix}2-\lambda & -2\\-2 & 2-\lambda\end{vmatrix} = (2-\lambda)^2 - 4 = (2-\lambda+2)(2-\lambda-2) = -\lambda(4-\lambda)$.
(difference of squares)

Eigenvalues are $\lambda=0$ and $\lambda=4$.

6b. $H = \begin{pmatrix}2 & -2\\-2 & 2\end{pmatrix} \sim \begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}$. Rank H = dim Col H = 1.

7a $MX = \lambda X$? Check: $\begin{pmatrix} 5 & -2 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & -1 & 3 & -6 \end{pmatrix} = \begin{pmatrix} 3 & 7 & -1 & 2 \\ 3 & 9 & -3 & 6 \end{pmatrix}$.

That is, $M\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$, $M\begin{pmatrix}1\\3\end{pmatrix} = -\begin{pmatrix}1\\3\end{pmatrix}$, and $M\begin{pmatrix}-2\\-6\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix}$.

$\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector w/ $\lambda = 3$. $\begin{pmatrix}1\\3\end{pmatrix}$ & $\begin{pmatrix}-2\\-6\end{pmatrix}$ are eigenvectors, $\lambda = -1$.

7b. Two distinct eigenvalues $\Rightarrow$ M is diagonalizable.
$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$. $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

8a. $(A \vdots c) =$

$$\left(\begin{array}{ccccc:c} 2 & 4 & 0 & -4 & 6 & -8 \\ -1 & -1 & -1 & 1 & -2 & 2 \\ 0 & -3 & 3 & 4 & -3 & 7 \end{array}\right) \sim \begin{pmatrix} 1 & 2 & 0 & -2 & 3 & -4 \\ -1 & -1 & -1 & 1 & -2 & 2 \\ 0 & -3 & 3 & 4 & -3 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -2 & 3 & -4 \\ 0 & 1 & -1 & -1 & 1 & -2 \\ 0 & -3 & 3 & 4 & -3 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & -2 & 3 & -4 \\ 0 & 1 & -1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \left(\begin{array}{ccccc:c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

(O = pivot)

x_3, x_5 free. $x_1 = -2x_3 - x_5$; $x_2 = -1 + x_3 - x_5$; $x_4 = 1$

$$X = \begin{pmatrix} -2x_3 - x_5 \\ -1 + x_3 - x_5 \\ x_3 \\ 1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

8b. Sol'n to $AX = 0$ is $x_3\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

8c. A has a pivot in each row, so, yes, col.s of A span $\mathbb{R}^3$.

8d. A fails to have a pivot in each column, so col.s of A are <u>not</u> linearly independent.

8e. T fails to be one-to-one for any of the following reasons.
i. Col.s of A fail to be linearly independent (8d)
ii. $T(x) = \begin{pmatrix} -8 \\ 2 \\ 7 \end{pmatrix}$ has more than one solution (8a)
iii. $T(x) = 0$ has more than one solution (8b)

8f. T $\underline{\text{is}}$ onto for either of the following.

i. Cols of A span $\mathbb{R}^3$

ii. $T(x) = b$ is consistent for any $b \in \mathbb{R}^3$.

9a $(G \,\vdots\, I) =$

$$\begin{pmatrix} 0 & 2 & -2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 3 & -3 & 4 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 0 & -4 & -1 \\ 0 & 2 & 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & 0 & 1/2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix}$$

$$G^{-1} = \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$

9b. $XH = G \Rightarrow XHH^{-1} = X = GH^{-1} = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 1 & -1 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -6 & -2 \\ -2 & -3 & -1 \\ 7 & 11 & 4 \end{pmatrix}.$

9c. $(GH)^{-1} = H^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1 & 0 \\ 1/2 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1 & 0 \\ -1/2 & -3 & -1 \\ 1/2 & 8 & 3 \end{pmatrix}.$

10a. Cu DNE b. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 6 \end{pmatrix}$. c. BC DNE d. $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} (0 \;\, 3) = \begin{pmatrix} 0 & 0 \\ 0 & 3 \\ 0 & 6 \end{pmatrix}$

e $v^T w \; (= v \cdot w) = (0 \;\, 3) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3.$ f. $= -101(CB) = \begin{pmatrix} -202 & 0 \\ -101 & -303 \\ 0 & -606 \end{pmatrix}.$

g. $B^T C^T = (CB)^T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 6 \end{pmatrix}$. h. $\sqrt{4+1} = \sqrt{5}$. i $= |-101| \, \|w\| = 101\sqrt{5}$.

11a. $\{(3\,q\,1\,r\,s), (0\,0\,2\,t\,u)\}$. b. Row R = Row P. c. $\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$. d. $\left\{ \begin{pmatrix} a \\ f \\ k \end{pmatrix} \begin{pmatrix} c \\ h \\ m \end{pmatrix} \right\}$.

e. # free vars = 3. f. Nul R = Nul P, so dim = 3.

12a. $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0.$

12b. $\forall\, w \in V\ \exists\, c_1, c_2, c_3$ in $\mathbb{R}$ so that $c_1 v_1 + c_2 v_2 + c_3 v_3 = w$.

12c. coordinate vectors, using standard basis, are

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Pivot in each col. $\Rightarrow$ these vectors are lin. independent $\Rightarrow \{1, t^2-1, t^3+2\}$ are also lin. independent.

12d. Coordinate vectors $= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sim \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & -1 \end{matrix}$. $\boxed{\ } =$ pivot.

Pivot in each row $\Rightarrow$ these vectors span $\mathbb{R}^2 \Rightarrow \{t, 2t+1, t-1\}$ span $\mathbb{P}_1$.