4 $\square$ : from Chap. 4 5 $\square$ : from Chap. 5

MATH 203–02 (Kunkle), Exam 3
100 pts, 50 minutes

Name: ________________

Apr. 9, 2014



No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

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5 $\boxed{1.}$ Let $E = \begin{pmatrix} 0 & -5 \\ 1 & 2 \end{pmatrix}$.

a (9 pts). Find all eigenvalues of $E$.

b (16 pts). Find $2 \times 2$ matrices $P$ and $C$ so that $E = PCP^{-1}$ and the transformation $\mathbf{x} \mapsto C\mathbf{x}$ is the composition of a rotation and a scaling.

4 $\boxed{2}$ (10 pts). The set $\mathcal{B} = \{1, 3-t, 2+t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinates of $\mathbf{q}(t) = 5t^2 - 1$ relative to $\mathcal{B}$.

4 $\boxed{3}$ (12 pts). Answer either part a or part b. Clearly indicate which one you're solving.

a. Let $J = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 - 2x_1x_2 = 0 \right\}$. Show that $J$ is not a subspace of $\mathbb{R}^2$.

b. Let $K = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; a - c = 2b \right\}$. Show that $K$ is a subspace of $\mathbb{R}^3$.

4 $\boxed{4.}$ Let

$$A = \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & 2 & -2 & 4 \\ 2 & 3 & 11 & -8 \\ -1 & 1 & -8 & 1 \end{pmatrix}.$$

a (12 pts). Find a basis for $\operatorname{Col} A$.

b (6 pts). Find a basis for $\operatorname{Row} A$.

c (4 pts). Give a basis for the vector space spanned by the four polynomials $1 + 2t^2 - t^3$, $3 + 2t + 3t^2 + t^3$, $4 - 2t + 11t^2 - 8t^3$, and $-1 + 4t - 8t^2 + t^3$.

d (2 pts). What was the dimension of the vector space in part c?

4 $\boxed{5.}$ Suppose $F$ is a $3 \times 11$ matrix.

a (2 pts). What is the largest possible rank of $F$?

b (5 pts). If $\operatorname{rank} F = 2$, what is the dimension of the space of all solutions to $F\mathbf{x} = \mathbf{0}$?

5 $\boxed{6a}$ (9 pts). Find all eigenvalues of $H = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 5 & 0 \\ -1 & 1 & 2 \end{pmatrix}$.

5 $\boxed{6b}$ (13 pts). Determine whether or not $H$ is diagonalizable.

1a. $|E-\lambda I| = \begin{vmatrix} -\lambda & -5 \\ 1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda)+5 = \lambda^2 - 2\lambda + 5 = 0$

Can solve by completing the square (or quad. formula):

$\lambda^2 - 2\lambda + 1 = -5+1$. $(\lambda-1)^2 = -4$. $\lambda - 1 = \pm 2i$. $\lambda = 1 \pm 2i$

1b. Can proceed with either eigenvalue.
Choose $\lambda = 1-2i = a - bi \Rightarrow a=1, b=2$. $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

Find eigenvector for $\lambda = 1-2i$.

$E - (1-2i)I = \begin{pmatrix} -1+2i & -5 \\ 1 & 2-(1-2i) \end{pmatrix} = \begin{pmatrix} -1+2i & -5 \\ 1 & 1+2i \end{pmatrix}$.

This matrix must have exactly one free variable, so it is row equivalent to $\begin{pmatrix} 1 & 1+2i \\ 0 & 0 \end{pmatrix}$. $x \in$ Nul space $\Rightarrow$

$x = \begin{pmatrix} -(1+2i)x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1-2i \\ 1 \end{pmatrix}$. $P = [\operatorname{Re} u, \operatorname{Im} u] = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$

(If we had used $\lambda = 1+2i$, $P = \begin{pmatrix} -1 & +2 \\ 1 & 0 \end{pmatrix}$.)

2.

Seek $[q]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ so that $a \cdot 1 + b(3-t) + c(2+t+t^2) = -1+5t^2$.

Equating coefficients of $1, t, t^2$ yields $\begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$

Augment, row reduce:

$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right) \sim \begin{pmatrix} 1 & 3 & 0 & -11 \\ 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -26 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 5 \end{pmatrix}$. $[q]_B = \begin{pmatrix} -26 \\ 5 \\ 5 \end{pmatrix}$.

alt form:

$\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -16 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5 \end{pmatrix}$. $[q]_B = \begin{pmatrix} -6 \\ -5 \\ 5 \end{pmatrix}$.

3a. (It's helpful to note that $x \in J$ iff $x_1 - 2x_1x_2 = 0$, or $x_1(1-2x_2) = 0$. That is, iff $x_1 = 0$ or $x_2 = 1/2$.)

$J$ is not closed under $+$. To see this, note $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ are in $J$, but their sum, $\begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$, is not. Therefore $J$ is not a subspace.

(As a matter of fact, $J$ is not closed under scalar multiplication: $\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \in J$, but $2\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is not.)

3b. $K$ is the nullspace of the matrix $[1 \ -2 \ -1]$,* and so is a subspace of $\mathbb{R}^3$.

*That's because $a - c = 2b$ iff $a - 2b - c = 0$.

If you didn't notice this, you could still prove $K$ is subspace the longer way. Here's how:

i. $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in $K$, since $0 - 0 = 2 \cdot 0$.

ii. Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ are in $K$, so that $a - c = 2b$ and $\alpha - 2\gamma = 2\beta$. Their sum $\begin{pmatrix} a+\alpha \\ b+\beta \\ c+\gamma \end{pmatrix}$ is in $K$ because

$$(a+\alpha) - (c+\gamma) = (a-c) + (\alpha - \gamma) = 2b + 2\beta = 2(b+\beta).$$

iii Suppose $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in K$ and $e$ is any scalar. Then $e\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ea \\ eb \\ ec \end{pmatrix} \in K$ because

$$ea - ec = e(a-c) = e(2b) = 2(eb).$$

4. $A = \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & 2 & -2 & 4 \\ 2 & 3 & 11 & -8 \\ -1 & 1 & -8 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -3 & 3 & -6 \\ 0 & 4 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = EF(A)$

a. Basis Col $A = \{$pivot cols of $A\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -8 \\ 1 \end{pmatrix} \right\}$

b. Basis Row $A = \{$pivot rows EFA$\} = \{ (1\ \ 3\ \ 4\ \ -1),$
$(0\ \ 1\ \ -1\ \ 2), (0\ \ 0\ \ 0\ \ 1) \}.$

c. Coefficients of these polynomials = columns of $A$.
Basis for their span "=" basis for Col $A$ :
$\{ 1 + 2t^2 - t^3,\ 3 + 2t + 3t^2 + t^3,\ -1 + 4t - 8t^2 + t^3 \}$

d. dimension = number of elements in a basis = 3.

5a. Rank $F$ = dim Row $F$ = dim Col $F$, cannot exceed number of rows or columns. If $F$ is $3 \times 11$, max rank = 3.
(alt: $F$ $3 \times 12$. max rank = 3.)

5b. dim Nul $F$ + rank $F$ = # columns = 11.
If rank $F = 2$, then dim Nul $F = 9$.
(alt: if rank $F = 2$, then dim Nul $F = 12 - 2 = 10$.)

6a. $|H - \lambda I| = \begin{vmatrix} -\lambda & 3 & 0 \\ -2 & 5-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix}$ $\begin{pmatrix}\text{expand along} \\ \text{column } 3\end{pmatrix}$ $= (2-\lambda)\big(-\lambda(5-\lambda) + 6\big)$

$= (2-\lambda)(\lambda^2 - 5\lambda + 6) = (2-\lambda)(\lambda-2)(\lambda-3) = -(\lambda-2)^2(\lambda-3)$
Eigenvalues are 2 and 3.

6b. $H$ diagonalizable iff geometric mult of $\lambda = 2$ is 2.
$H - 2I = \begin{pmatrix} -2 & 3 & 0 \\ -2 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

One free var $\Rightarrow$ dim Nul$(H - 2I) = 1$, not 2. $H$ <u>not</u> diagonalizable.