3$\square$: from Chap. 3   2$\square$: from Chap. 2

MATH 203–02 (Kunkle), Exam 2   Name: ______________________
100 pts, 50 minutes   Feb. 24, 2014

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

Solve always means to find the general solution, if it exists.

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3 $\boxed{1.}$ If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = -4$, find the following determinants.

a (6 pts). $\begin{vmatrix} a & x & d \\ b & y & e \\ c & z & f \end{vmatrix}$ b (6 pts). $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2x & 2y & 2z \end{vmatrix}$ c (6 pts). $\begin{vmatrix} 0 & 0 & c \\ 0 & 0 & f \\ z & y & x \end{vmatrix}$

2. Find the following (if it exists), if $A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

2 $\boxed{a}$ (16 pts) A^{-1} 3 $\boxed{b}$ (12 pts) $\det(A)$

2 $\boxed{3}$ (8 pts). Compute the product (if it exists).

a. $\begin{pmatrix} 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ b. $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} -2 & 5 \end{pmatrix}$ c. $\begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 & -1 \\ 0 & -1 & 1 \\ 3 & 5 & 0 \end{pmatrix}$

2 $\boxed{4}$ Suppose $X^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $Y^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, and $Z^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$.

a (4 pts). Solve for $\mathbf{u}$ (if it exists) in the equation $Z\mathbf{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

b (8 pts). Solve for C (if it exists) in the equation $CX = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & -1 \end{pmatrix}$.

c (8 pts). Find $(X^TY)^{-1}$(if it exists).

5 (8 pts). Answer **either** part a **or** part b. **Clearly indicate which question you're answering.**

3 $\boxed{a.}$ If A is invertible, explain why $\det(A^{-1}) = (\det A)^{-1}$.

2 $\boxed{b.}$ If B is a 4×4 matrix whose columns are linearly independent, explain why the columns of B^2 must span $\mathbb{R}^4$.

3 $\boxed{6.}$ Let $D = \begin{pmatrix} 2 & 0 & -5 \\ 0 & a & 7 \\ 3 & 1 & 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Suppose that $\det(D) = \frac{1}{3}$.

a (9 pts). If $D\mathbf{x} = \mathbf{c}$, find x_2.   b (9 pts). Find $(D^{-1})_{3,2}$

1.a. $\begin{vmatrix} a & x & d \\ b & y & e \\ c & z & f \end{vmatrix} \underset{\text{transpose}}{=} \begin{vmatrix} a & b & c \\ x & y & z \\ d & e & f \end{vmatrix} \underset{\text{row interchange.}}{=} - \begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = -(-4) = 4$ (alt 5)

b. $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2x & 2y & 2z \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 2x & 2y & 2z \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} a & b & c \\ d & e & f \\ 2x & 2y & 2z \end{vmatrix} = 2 \cdot 2 \cdot 2 \begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = -32$ (alt -40)

c. $\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & f \\ z & y & x \end{pmatrix}$ is singular because rows 1 & 2 can't both have a pivot. Determinant $= 0$.

2. $\begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{a.}{\sim} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 0 & 1 \end{pmatrix}$

$\overset{b.}{\sim} \begin{pmatrix} 1 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1/4 & 0 & 1/4 \end{pmatrix} \overset{c.}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 1/4 & 0 & 1/4 \end{pmatrix}$

Steps

a. $r_3 \leftarrow r_3 - r_1$

$r_4 \leftarrow r_4 + r_2$

b. $r_2 \leftrightarrow r_3$, then

$r_2 \leftarrow -r_2$

$r_4 \leftarrow \frac{1}{4} r_4$

c. $r_3 \leftarrow r_3 - 2r_4$

$r_1 \leftarrow r_1 - 3r_2$

a. $A^{-1} = \begin{pmatrix} -2 & 0 & 3 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 1/4 & 0 & 1/4 \end{pmatrix}$

b. $|A| = \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -(-4) = 4.$

2 alt: a. $\left(\begin{array}{cccc|cccc} 1&3&0&0&1&0&0&0\\ 0&0&1&2&0&1&0&0\\ 1&2&0&0&0&0&1&0\\ 0&0&1&-2&0&0&0&1 \end{array}\right) \overset{1.}{\sim} \left(\begin{array}{cccc|cccc} 1&3&0&0&1&0&0&0\\ 1&2&0&0&0&0&1&0\\ 0&0&1&2&0&1&0&0\\ 0&0&1&-2&0&0&0&1 \end{array}\right)$

$\overset{2.}{\sim} \left(\begin{array}{cccc|cccc} 1&3&0&0&1&0&0&0\\ 0&-1&0&0&-1&0&1&0\\ 0&0&1&2&0&1&0&0\\ 0&0&0&-4&0&-1&0&1 \end{array}\right) \overset{3.}{\sim} \left(\begin{array}{cccc|cccc} 1&0&0&0&-2&0&3&0\\ 0&1&0&0&1&0&-1&0\\ 0&0&1&0&0&1/2&0&1/2\\ 0&0&0&1&0&1/4&0&-1/4 \end{array}\right). \quad A^{-1} = \begin{pmatrix} -2&0&3&0\\ 1&0&-1&0\\ 0&1/2&0&1/2\\ 0&1/4&0&-1/4 \end{pmatrix}$

steps: 1. $r_2 \leftrightarrow r_3$ 2. $r_4 \leftarrow r_4 - r_3$, $r_2 \leftarrow r_2 - r_1$. 3. $r_4 \leftarrow r_4 \div (-4)$, $r_2 \leftarrow -1 \cdot r_2$,

(3. continued) then $r_3 \leftarrow r_3 - 2r_4$, $r_1 \leftarrow r_1 - 3r_2$.

b. $\det A = - \begin{vmatrix} 1&3&0&0\\ 1&2&0&0\\ 0&0&1&2\\ 0&0&1&-2 \end{vmatrix} = - \begin{vmatrix} 1&3&0&0\\ 0&-1&0&0\\ 0&0&1&2\\ 0&0&0&-4 \end{vmatrix} = -(1)(-1)(1)(-4) = -4.$

one row interchange to here

3a. $\underset{(1\times3)}{\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}} \underset{(3\times1)}{\begin{pmatrix} 1\\ -1\\ 5 \end{pmatrix}} = \underset{(1\times1)}{1\cdot1 + 2\cdot(-1) + (-3)\cdot5 = -16}.$ alt: $\begin{pmatrix} 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 1\\ -1\\ 5 \end{pmatrix} = 1-3-10 = -12.$

b. $\underset{3\times1}{\begin{pmatrix} 1\\ 0\\ -2 \end{pmatrix}} \underset{1\times2}{\begin{pmatrix} -2 & 5 \end{pmatrix}} = \underset{3\times2}{\begin{pmatrix} -2 & 5\\ 0 & 0\\ +4 & -10 \end{pmatrix}}.$ alt: $\begin{pmatrix} 1\\ 0\\ -2 \end{pmatrix}\begin{pmatrix} 2 & -5 \end{pmatrix} = \begin{pmatrix} 2 & -5\\ 0 & 0\\ -4 & 10 \end{pmatrix}.$

c. DNE. can't multiply 3×2 by 3×3.

4. It is not necessary to find X, Y, or Z to solve this problem.

a. $u = Z^{-1}\begin{pmatrix} 3\\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1\\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3\\ 2 \end{pmatrix} = \begin{pmatrix} 1\\ 6 \end{pmatrix}.$ alt. $Z^{-1}\begin{pmatrix} 2\\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1\\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2\\ 3 \end{pmatrix} = \begin{pmatrix} -1\\ 9 \end{pmatrix}.$

b. $C = \begin{pmatrix} 1 & 0\\ -1 & 2\\ 0 & -1 \end{pmatrix} X^{-1} = \begin{pmatrix} 1 & 0\\ -1 & 2\\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0\\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0\\ -3 & 2\\ 1 & -1 \end{pmatrix}.$ alt. $C = \begin{pmatrix} 0 & 1\\ 2 & -1\\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0\\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1\\ 3 & -1\\ -1 & 0 \end{pmatrix}.$

c. $(X^T Y)^{-1} = Y^{-1}(X^T)^{-1} = Y^{-1}(X^{-1})^T = \begin{pmatrix} 0 & 1\\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1\\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 2 & -2 \end{pmatrix}.$

5a. $AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I) = 1$. The determinant of the product of 2 square matrices is the product of their determinants, so $\det(A)\cdot\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \dfrac{1}{\det A} = (\det A)^{-1}$.

5b. Cols of B linearly independent $\Rightarrow$ B invertible, by Invertible Matrix Theorem. Because the product of invertible matrices is invertible, $B^2 = BB$ must also be invertible. By IVT again, col.s of B^2 must span $\mathbb{R}^4$.

6a. Cramer's Rule $\Rightarrow$ $x_2 = \begin{vmatrix} 2 & 1 & -5 \\ 0 & 0 & 7 \\ 3 & -1 & 0 \end{vmatrix} \div \det(D)$. Expand along row 2:

$x_2 = -7\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \div \frac{1}{3} = -7(-2-3)\cdot 3 = 105$. Alt: $x_2 = -7(-2-3)\cdot 5 = 175$.

b. $D^{-1} = \dfrac{1}{\det D}(\text{cofactors of } D)^T$, so $(D^{-1})_{3,2} = \dfrac{1}{1/3}(\underset{\text{row}}{2},\underset{\text{col}}{3}$ cofactor of D$)$

$$\begin{array}{ccc} 2 & 0 & \cancel{-5} \\ \cancel{0} & \cancel{0} & \cancel{7} \\ 3 & 1 & \cancel{0} \end{array}$$

$= \dfrac{1}{1/3}(-1)^{2+3}\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 3(-1)(2) = -6.$

alt: $D^{-1}_{32} = 5(-1)(2) = -10.$