MATH 203–02 (Kunkle), Exam 1 Name: ______________________

100 pts, 50 minutes Jan. 31, 2014

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

Solve always means to find the general solution, if it exists.

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1 (10 pts). Compute the product, if it exists.

a. $\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ b. $\begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

2 (8 pts). Answer **either** part a **or** part b. **Clearly indicate which question you're answering.** In both parts, A is a matrix, and $\mathbf{b}$, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{v}$ are vectors of the appropriate size.

a. Suppose that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$. Prove that $\mathbf{p} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

b. Suppose $\mathbf{p}$ and $\mathbf{q}$ are both solutions to $A\mathbf{x} = \mathbf{b}$. Prove that $\mathbf{p} - \mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

3. Let V be the set of vectors $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \\ 5 \end{pmatrix} \right\}$.

a (20 pts). Determine whether $\begin{pmatrix} -1 \\ 3 \\ -1 \\ 4 \end{pmatrix}$ is in the span of V.

b (5 pts). Is V linearly independent? Briefly explain.

c (4 pts). State the general solution to: $x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -4 \\ 3 \\ 5 \end{pmatrix} = \mathbf{0}$

4 (24 pts). Let $A = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 5 & 4 \\ 3 & -2 & 11 & 4 \end{pmatrix}$. Solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$. Write your answer in parametric vector form (or as a vector, if only one solution exists).

5 (5 pts). Do the columns of A in Problem 4 span $\mathbb{R}^3$? Briefly explain.

6. Let $S : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the rule $S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + 4x_4 \\ x_2 - 3x_3 \end{pmatrix}$.

a (6 pts). Find the standard matrix for the transformation S.

b (6 pts). Define what it means for a transformation T to be one-to-one.

c (6 pts). Define what it means for a transformation T to map $\mathbb{R}^4$ onto $\mathbb{R}^2$.

d (6 pts). Does S map $\mathbb{R}^4$ onto $\mathbb{R}^2$? Briefly explain.

Several problems on this exam appeared in two forms

1. a. $\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ is $2\times \underbrace{3 \text{ times } 2}_{\neq}\times 1$. DNE

b. $\begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$.

alternate form:

a. $\begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = (-2)\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 1\begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -1 \end{pmatrix}$.

b. $\begin{pmatrix} 1 & -3 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ is $3\times \underbrace{2 \text{ times } 3}_{\neq}\times 1$. DNE

2a. Given $Ap = b$, $Av = 0$, need show $A(p+v) = b$:

$$A(p+v) = Ap + Av \quad \text{by distributive law (Thm 5, p.45)}$$
$$= b + 0 = b \checkmark$$

2b. Given $Ap = b$, $Aq = b$, need show $A(p-q) = 0$:

$$A(p-q) = Ap - Aq \quad \text{(Thm 5, p.45)}$$
$$= b - b = 0 \checkmark$$

3. Augment and row reduce to echelon form:

$$\begin{pmatrix} 1 & 3 & 2 & -1 \\ 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ -2 & 3 & 5 & 4 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 1 & 3 & 2 & -1 \\ 0 & 1 & 3 & -1 \\ -2 & 3 & 5 & 4 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 3 & 6 & -4 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & -3 & 10 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & 6 & -4 \\ 0 & 3 & -3 & 10 \end{pmatrix}$$

$$\xrightarrow{4} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -12 & 13 \end{pmatrix} \xrightarrow{5} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 17 \end{pmatrix}.$$

steps. 1: $r_1 \leftrightarrow r_2$ 2: $r_2 \leftarrow r_2 - r_1$, $r_4 \leftarrow r_4 + 2r_1$

3: $r_2 \leftrightarrow r_3$ 4: $r_3 \leftarrow r_3 - 3\cdot r_2$, $r_4 \leftarrow r_4 - 3r_2$

5: $r_4 \leftarrow r_4 - 4r_3$

a. O = pivots. Pivot in last column means system inconsistent, so $\begin{pmatrix} -1 \\ 3 \\ -1 \\ 4 \end{pmatrix}$ is NOT in span V.

b. V has a pivot in each column. (No free variables) V is linearly independent.

c. For reasons in b., only solution is $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

4. (To show you both forms of this question, I need to solve $Ax = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$ and $Ax = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix}$. I'll augment A with both and then row reduce.)

$$\left(\begin{array}{cccc|c|c} 1 & 0 & 3 & 2 & 1 & 3 \\ 2 & 1 & 5 & 4 & 6 & 8 \\ 3 & -2 & 11 & 4 & -4 & 6 \end{array}\right) \xrightarrow{1} \left(\begin{array}{cccc|c|c} 1 & 0 & 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 0 & 4 & 2 \\ 0 & -2 & 2 & -2 & -7 & -3 \end{array}\right) \xrightarrow{2} \left(\begin{array}{cccc|c|c} 1 & 0 & 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{3} \left(\begin{array}{cccc|c|c} 1 & 0 & 3 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right) \xrightarrow{4} \left(\begin{array}{cccc|c|c} 1 & 0 & 3 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -1/2 & -1/2 \end{array}\right)$$

Steps. $\underbrace{1: r_2 \leftarrow r_2 - 2r_1,\ r_3 \leftarrow r_3 - 3r_1. \quad 2: r_3 \leftarrow r_3 + 2r_1}_{\text{(FORWARD PHASE)}}$ $\quad \underbrace{3: r_1 \leftarrow r_1 + r_3 \quad 4: r_3 \leftarrow r_3 \div (-2)}_{\text{(BACKWARD PHASE)}}$

Solution: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 3x_3 \\ 4 + x_3 \\ x_3 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ -1/2 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. (in case $Ax = \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$)

or $x = \begin{pmatrix} 4 - 3x_3 \\ 2 + x_3 \\ x_3 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ -1/2 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (in case $Ax = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix}$).

5. A has a pivot in every row, so columns of A span $\mathbb{R}^3$.

6. a. $S(x) = \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & -3 & 0 \end{pmatrix} x$, so standard matrix $= \begin{pmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & -3 & 0 \end{pmatrix} = A$.

○ = pivots

b. T one-to-one means $T(x) = T(y) \Rightarrow x = y$. In other words, no b in range is the image under T of more than one x in the domain.

c. T onto $\mathbb{R}^2$ means that for every b in $\mathbb{R}^2$, there's an x in $\mathbb{R}^4$ for which $T(x) = b$. In other words, the range of T is all of $\mathbb{R}^2$.

d. (There were two forms of part d. Here are both answers)
A has a pivot in every row, so columns of A span $\mathbb{R}^2$, so A <u>is</u> onto.

alt: A fails to have a pivot in every column. Cols. of A are linearly dependent, so A is <u>not</u> one-to-one.