1. Sketch the graph of a function $q(x)$ that satisfies all of the following:
   - $q'(x) < 0$ for all $x \neq -1$.
   - $x = -1$ is a vertical asymptote to the curve $y = q(x)$.
   - $q''(x) > 0$ if $-1 < x < 2$.
   - $q''(x) < 0$ if $x < -1$ or $x > 2$.

2. Find all critical points of $f(x) = x^{2/3}(x - 2)$.

3. Let $g(x) = 4x + \frac{1}{x}$.
   a. Find all $x$-values, if any, at which $g(x)$ is a local maximum.
   b. Find all $x$-values, if any, at which $g(x)$ is a local minimum.

   You are not required to tell me these maximum or minimum values of $g(x)$.

4a. On what interval(s) is $h(x) = 4x^3 - 3x^2 - 6x$ decreasing? On what interval(s) is $y = h(x)$ concave down?

4b. At which $x$-values, if any, does $h(x)$ have an inflection point?

4c. Find the absolute maximum and absolute minimum values of $h(x)$ on the interval $[-2, 0]$.

5. Evaluate the following limits.
   a. $\lim_{x \to 0} \frac{\cos(3x) - \cos(5x)}{x^2}$
   b. $\lim_{x \to \infty} x^{1/3}e^{-2x}$

6. Find the equations of all horizontal and vertical asymptotes of the curve.
   a. $y = x^{1/3}e^{-2x}$
   b. $y = \frac{2x^2 + 1}{x^2 - 16}$

7. Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem as applied to the function $p(x) = x - 2\sin x$ on the interval $[0, \pi]$. 
1. (Source: 4.3.28) The graph at the right in one correct solution, but there are others. For instance, \( y = q(x) \) could have a horizontal asymptote as \( x \to -\infty \).

\( q(x) \) is always decreasing since \( q'(x) \) is always negative (when it exists). The graph could have a horizontal asymptote to the left, but nothing in the problem says that it must. There’s an inflection point at \( x = 2 \). Note that the tangent line at \( x = 2 \) (which you were not required to include in your graph) crosses the curve. This is because \( q'(x) \) has a local max at \( x = 2 \).

The function \( q(x) = \frac{1}{x+1} - e^{(x-4.6)} \) (approximately) has a graph as described in this problem, as does \( \frac{1}{x+1} - e^{(x-4.6)} - ax \) for any positive number \( a \).

2. (Source: 4.1.39) Critical points are those \( x \)-values at which \( f(x) \) is defined but \( f'(x) \) is either zero or undefined, so we calculate \( f'(x) = (x^{2/3}(x-2))' = \frac{2}{3}x^{-1/3}(x-2) + x^{2/3} \). The negative exponent means that \( f' \) does not exist at \( x = 0 \). To find the zeros of \( f' \), it helps to factor \( f'(x) \):

\[
x^{-1/3}\left(\frac{2}{3}(x-2)+x\right) = x^{-1/3}\left(\frac{5}{3}x-\frac{4}{3}\right) = 0 \Rightarrow x^{-1/3} = 0 \text{ (no solutions)} \text{ or } \frac{5}{3}x-\frac{4}{3} = 0 \Rightarrow x = \frac{4}{5}
\]

So the critical points are \( x = 0 \) and \( x = \frac{4}{5} \).

(You could solve the equation \( f'(x) = 0 \) without factoring. Below, I multiplied both sides of the equation by \( 3x^{1/3} \):

\[
\frac{2}{3}x^{-1/3}(x-2) + x^{2/3} = 0 \Rightarrow 0 = 2(x-2) + 3x = 5x - 4 \Rightarrow x = \frac{4}{5} \text{ again.}
\]

3. (Source: 4.1.53, 4.3.12) Local extrema of \( g(x) = 4x + \frac{1}{x} = 4x + x^{-1} \) can occur only at critical points, i.e., those points where \( g'(x) = 4 - x^{-2} = 4 - \frac{1}{x^2} \) is undefined or zero. \( g' \) fails to exist only at \( x = 0 \), where \( g \) is also undefined, so local extrema can only occur where

\[
4 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1/2.
\]

Apply the second derivative test at these points: \( g''(x) = 2x^{-3} \), so \( g(1/2) = (1/2)^{-3} = (2)^3 = 8 \) and \( g(-1/2) = (-1/2)^{-3} = (-2)^3 = -8 \). Because \( g \) is concave up at \( x = 1/2 \), \( g(1/2) \) is a local minimum, and because \( g \) is concave down at \( x = -1/2 \), \( g(-1/2) \) is a local maximum. (As explained in the statement of the Problem 3, you were not required to calculate these \( g \)-values.)
4a. (Source: 4.3.9) Make a sign chart for 
\[ h'(x) = 12x^2 - 6x - 6 = 6(2x + 1)(x - 1) \] and 
\[ h''(x) = 24x - 6 = 6(4x - 1). \]

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\[ x : - \frac{1}{2} \quad \frac{1}{4} \quad 1 \]

\[ h \text{ decreases when } h'(x) < 0, \text{ on } (-\frac{1}{2}, 1) \text{ and is concave down when } h''(x) < 0, \text{ on } (-\infty, \frac{1}{4}). \]

4b. \( h(x) \) has an inflection point at \( x = \frac{1}{4} \), where \( h''(x) \) changes sign.

4c. (Source: 4.1.51) The absolute extrema can only occur at an endpoint or a critical point inside \((-2, 0)\), so just test \( h(x) \) at \( x = -2, -1/2, \) and 0:
\[ h(-2) = -32 \quad h(-1/2) = 7/4 \quad h(0) = 0 \]
The absolute maximum is 7/4 and the minimum is -32.

5a. (Source: 4.4.34) \( \lim_{x \to 0} \frac{\cos(3x) - \cos(5x)}{x^2} = \frac{\pi}{0} \). l'Hôpital: \( \lim_{x \to 0} \frac{-3\sin(3x) + 5\sin(5x)}{2x} = \frac{-9 + 25}{2} = 8 \), so, by two applications of l'Hôpital’s Rule, the original limit also equals 8.

5b. (Source: 4.4.44, 47) Write the product as a quotient: \( \lim_{x \to \infty} x^{1/3} e^{-2x} = \lim_{x \to \infty} \frac{x^{1/3}}{e^{2x}} = \frac{\infty}{\infty} \), so we can try l'Hôpital's: \( \lim_{x \to \infty} \frac{\frac{1}{3}x^{-2/3}}{2e^{2x}} = \lim_{x \to \infty} \frac{1}{6x^{2/3}e^{2x}} = \frac{1}{\infty} = 0 \), so the original limit must also equal 0.

6a. (Source: 4.3.53, 4.4.44, 4.5.43) By Problem 5b, \( y = 0 \) is a horizontal asymptote of \( y = x^{1/3} e^{-2x} \) as \( x \to \infty \). (As \( x \to -\infty \), \( x^{1/3} e^{-2x} \to -\infty \cdot \infty = -\infty \), so the curve as no horizontal asymptote on the left. \( y \) is continuous on \((-\infty, \infty)\) and therefore has no vertical asymptotes.)

6b. (Source: 4.5.11) As \( x \to \pm \infty \), \( y = \frac{2x^2 + 1}{x^2 - 16} \) has the same limit as \( y = \frac{2x^2}{x^2} = 2 \), so \( y = 2 \) is a horizontal asymptote.

Vertical asymptotes occur at \( x = 4 \) and \( x = -4 \), since at these \( x \)-values, \( \frac{2x^2 + 1}{x^2 - 16} \) is non-zero, indicating that \( y \to \pm \infty \).

7. (Source: 4.2.7) Seek \( c \) in \((0, \pi)\) for which
\[ p'(c) = 1 - 2 \cos c = \frac{p(\pi) - p(0)}{\pi - 0} = \frac{(\pi - 2 \sin \pi) - (0 - 2 \sin 0)}{\pi} = \frac{(\pi - 0) - (0)}{\pi} = 1, \]
or \( \cos c = 0 \). The only solution on \((0, \pi)\) is \( c = \pi/2 \).