MATH 120–03 (Kunkle), Exam 2 Name: ______________________

100 pts, 75 minutes Oct 20, 2016

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

1a (14 pts). Find $\frac{dy}{dx}$ along the curve $\sin(x^2y) = x^3 - y^3 - 1$

1b (6 pts). Find an equation of the line tangent to the curve in 1a at the point $(1, 0)$.

2. Differentiate the given function. You are not required to simplify your answers.

a (6 pts). $2\ln x + 4\tan^{-1} x - e^x \sec^{-1} x$ b (7 pts). $\frac{\cos x}{2+\cot x}$ c (6 pts). $5x(\tan x)(\sin^{-1} x)$

d (3 pts). $\csc^4 x$ e (9 pts). $(\tan x)^{1/x}$ f (5 pts). $\cos\sqrt{\sec(\tan x)}$

3. An object moving along a coordinate axis is at position $p(t) = (t^2 - 4)^2$ meters at time t seconds. Assume that $-\infty < t < \infty$.

Work in the space below and label your answers to the following questions.

a (6 pts). Express the object's velocity and acceleration as functions of t. Label your answers and include the correct units.

b (3 pts). At what time(s), if any, is velocity zero?

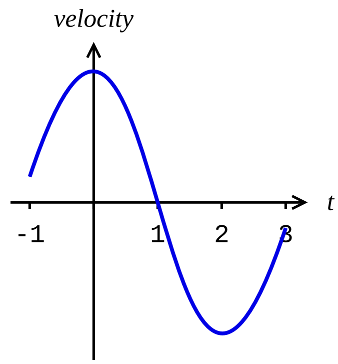
c (4 pts). Over what interval(s) of time is the object moving in the positive direction?

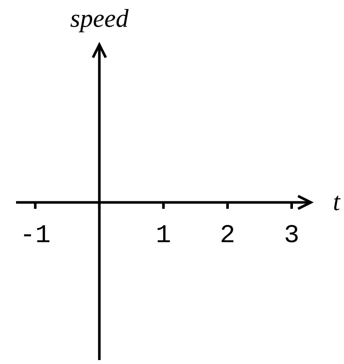
d (6 pts). Find the total distance the object travels between times $t = -2$ and $t = 3$. (Although not required, a schematic diagram might help you visualize the object's motion over this interval of time.)

4. An object moves along a coordinate axis from time $t = -1$ to $t = 3$. The figure on the left shows the graph of the object's velocity.

a (2 pts). On the axes provided, sketch the graph of the object's speed.

b (3 pts). Over what interval(s) of time is the object speeding up?





5a (6 pts). Find the linearization $L(x)$ of the function $r(x) = (x - 2)^3$ at $x = 1$.

5bs (2 pts). Find the differential dr, where r is the function in 5a.

6 (12 pts). A floodlight on the ground shines on a child walking away from the light and towards a wall. If the wall is 18 m from the light and the child is 1 m tall and walks in a straight line towards the wall at 1.4 m per second, how fast is the child's shadow on the wall shortening when she is 12 m from the wall? (Assume that the ground is horizontal and the wall is vertical.)

1a. (Source: 3.5.17) Think of y as an unspecified function of x, differentiate both sides of $\sin(x^2y) = x^3 - y^3 - 1$ with respect to x, and solve for $\frac{dy}{dx}$:

$$\cos(x^2y)\left(2xy + x^2\frac{dy}{dx}\right) = 3x^2 - 3y^2\frac{dy}{dx}$$

$$2xy\cos(x^2y) + \frac{dy}{dx}x^2\cos(x^2y) = 3x^2 - 3y^2\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2\cos(x^2y) = 3x^2 - 2xy\cos(x^2y)$$

$$\frac{dy}{dx}\left(3y^2 + x^2\cos(x^2y)\right) = 3x^2 - 2xy\cos(x^2y)$$

$$\frac{dy}{dx} = \frac{3x^2 - 2xy\cos(x^2y)}{3y^2 + x^2\cos(x^2y)}$$

1b. (Source: 3.5.25-32) At $(1, 0)$, we compute $\frac{dy}{dx} = \frac{3-0}{0+1\cos 0} = 3$, so the point-slope equation of the line is $y = 3(x - 1)$.

2a. (Source: 3.2.3, 3.5.49,53, 3.6.2) The last term requires the product rule.
$(2\ln x + 4\tan^{-1} x - e^x \sec^{-1} x)' =$
$2 \cdot \frac{1}{x} + 4 \cdot \frac{1}{x^2+1} - (e^x \sec^{-1} x + e^x \frac{1}{|x|\sqrt{x^2-1}}) = \frac{2}{x} + \frac{4}{x^2+1} - e^x(\sec^{-1} x + \frac{1}{|x|\sqrt{x^2-1}})$.

You were not required to simplify as I did above. Depending on how $\sec^{-1} x$ is defined, its derivative is either $\frac{1}{|x|\sqrt{x^2-1}}$ or $\frac{1}{x\sqrt{x^2-1}}$. I accepted either of these for full credit.

2b. (Source: 3.3.11) Quotient rule:
$\left(\frac{\cos x}{2+\cot x}\right)' = \frac{(\cos x)'(2+\cot x) - \cos x(2+\cot x)'}{(2+\cot x)^2} = \frac{-\sin x(2+\cot x)+\cos x(\csc^2 x)'}{(2+\cot x)^2}$.

2c. (Source: 3.3.15, 3.5.51)

$$\begin{aligned}&(5x)'(\tan x)(\sin^{-1} x) + (5x)(\tan x)'(\sin^{-1} x) + (5x)(\tan x)(\sin^{-1} x)'\\ &= 5(\tan x)(\sin^{-1} x) + 5x(\sec^2 x)(\sin^{-1} x) + 5x(\tan x)\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

2d. (Source: 3.3.4, 3.4.37) $(\csc^4 x)' = 4\csc^3 x(-\csc x \cot x) = -4\csc^4 x \cot x$

2e. (Source: 3.6.49) Either use logrithmic differentiation or, as shown here, rewrite $(\tan x)^{1/x}$ as $e^{\ln((\tan x)^{x^{-1}})} = e^{x^{-1}\ln(\tan x)}$. Now differentiate with the chain rule first, and then the product rule, and then the chain rule:

$$\begin{aligned}(e^{x^{-1}\ln(\tan x)})' &= e^{x^{-1}\ln(\tan x)}(x^{-1}\ln(\tan x))'\\ &= e^{x^{-1}\ln(\tan x)}(-x^{-2}\ln(\tan x) + x^{-1}(\ln(\tan x))')\\ &= e^{x^{-1}\ln(\tan x)}\left(-x^{-2}\ln(\tan x) + x^{-1}\frac{1}{\tan x}\cdot\sec^2 x\right)\end{aligned}$$

2f. (Source: 3.4.45) Several applications of the chain rule: $(\cos(\sec(\tan x))^{1/2})' =$
$-\sin((\sec(\tan x))^{1/2})\frac{1}{2}(\sec(\tan x))^{-1/2}\sec(\tan x)\tan(\tan x)\sec^2 x$

3a. (Source: 3.7.1) Position $= p = (t^2-4)^2$ m, velocity $= v = p' = 2(t^2-4)2t = 4t(t^2-4) = 4t^3 - 16t$ m/sec and acceleration $= a = p'' = 12t^2 - 16$ m/sec^2.

3b. $v = 2(t^2-4)2t = 4t(t-2)(t+2) = 0$ at $t = -2$, 0, and 2.

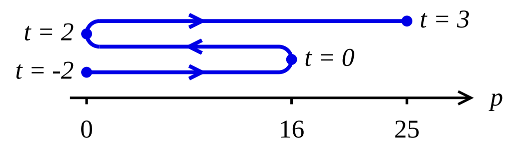
3c. The object moving in the positive direction when $v > 0$. Make a sign chart for v as in precalculus:

$$\begin{array}{rl} 4t(t-2)(t+2): & ----0++++0----0++++ \\ \hline t: & \qquad -2 \qquad\quad 0 \qquad\quad 2 \end{array}$$

So, the object is moving forward for t in $(-2, 0)$ and in $(2, \infty)$.

3d. The object moves forward from $t = -2$ to $t = 0$, then backward between $t = 0$ and $t = 2$, and then forward from $t = 2$ to $t = 3$. Calculating p at these times, we can picture the object's motion as in this schematic diagram:

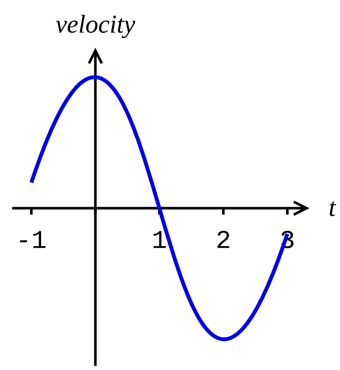
t	-2	0	2	3
$p(t)$	0	16	0	25

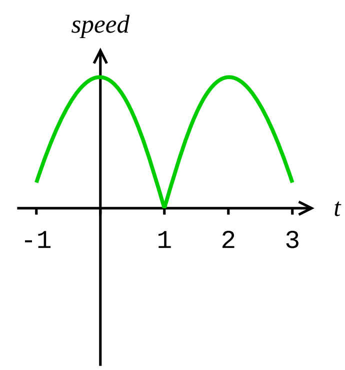


Therefore, the total distance traveled between $t = -2$ and $t = 3$ is $16 + 16 + 25 = 57$m.

4a. (Source: 3.7.5) speed $= |\text{velocity}|$. See graph at right.

4b. The object speeding up when speed is increasing. That is, when $-1 \le t \le 0$ and $1 \le t \le 2$.



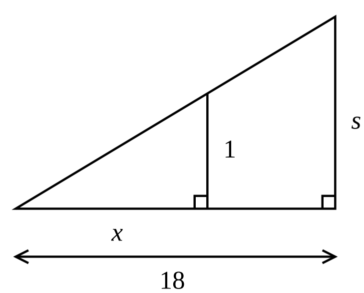


5a. (Source: 3.10.1) $r(1) = -1$, $r'(x) = 3(x-2)^2$, and $r'(1) = 3$, so the linearization $L(x) = r(1) + r'(1)(x-1) = -1 + 3(x-1)$.

5b. (Source: 3.10.11) $dr = \dfrac{dr}{dx}dx = 3(x-2)^2\,dx$.

6. (Source: 3.9.18) Let x be the distance from the child to the light, and s be the height of her shadow on the wall. We're given $\frac{dx}{dt} = 1.4$ and asked to find $\frac{ds}{dt}$. To find a relation between x and s, use similar triangles:

$$\frac{s}{18} = \frac{1}{x} = x^{-1}$$



Now differentiate with respect to time t:

$$\frac{ds}{dt}\frac{1}{18} = -x^{-2}\frac{dx}{dt}$$

We can solve for $\frac{ds}{dt}$ after plugging in the values of x and $\frac{dx}{dt}$. At the moment in question, $x = 6$, so

$$\frac{ds}{dt}\frac{1}{18} = -\frac{1}{6^2}(1.4) \Rightarrow \frac{ds}{dt} = -\frac{18}{36}(1.4) = -0.7\text{m/sec}.$$

That is, the child's shadow is shortening at the rate of 0.7 meters per second.