No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You will not receive credit for using l'Hospital's Rule (a technique learned later in this course) on any problem on this exam.

When a problem instructs you to "evaluate" a limit, the expected answer is ∞, a number, $-\infty$, or "does not exist." In any case, supporting work is required along with your answer.

You are required to use the definition of derivative as a limit only if the problem specifically says so. Otherwise, you can use any correct method to find a derivative.

1a (5 pts). State the precise, ε-δ definition of what it means for $\lim_{x\to a} f(x)$ to equal L.

1b (10 pts). Write an ε-δ proof of the fact that $\lim_{x\to 2}(5x-13) = -3$.

2. Evaluate the limit.

a (8 pts). $\displaystyle\lim_{x\to 2}\left(\frac{1}{2-x} - \frac{2}{2x-x^2}\right)$

b (5 pts). $\displaystyle\lim_{x\to -\infty}\frac{4x^5+6x}{2x^3-4x+5}$

c (7 pts). $\displaystyle\lim_{x\to -2}\frac{3x^2+2x-8}{x^2+5x+6}$

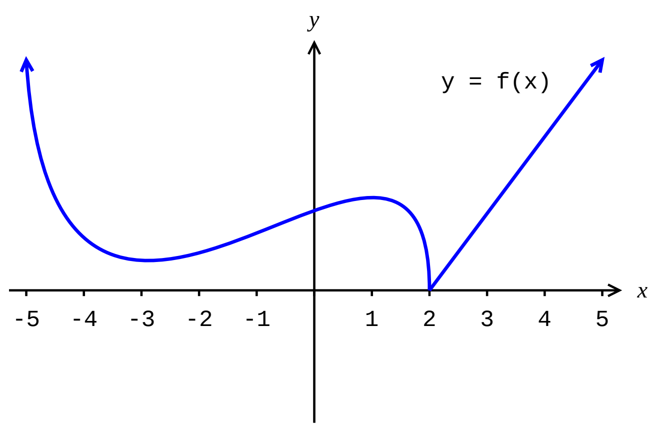
d (5 pts). $\displaystyle\lim_{x\to -3^+}\frac{3x^2+2x-8}{x^2+5x+6}$

e (5 pts). $\displaystyle\lim_{x\to -3^-}\frac{3x^2+2x-8}{x^2+5x+6}$

f (5 pts). $\displaystyle\lim_{x\to \infty}\frac{3x^2+2x-8}{x^2+5x+6}$

3 (4 pts). Give the equations of all horizontal and vertical asymptotes of $y = \frac{3x^2+2x-8}{x^2+5x+6}$. If your answers follow from your work on Problem 2, then no further supporting work will be required.

4 (10 pts). The graph of the function $f(x)$ appears in the figure at right. Sketch the graph of $f'(x)$ on the axes provided.



5 (15 pts). Let $s(x) = 3x^5 - \frac{2}{x^5} - 3\sqrt[3]{x^2} + e^x$. Use any correct method to find the following.

a. $s'(x)$ b. $s''(x)$ c. An equation of the line tangent to $y = s(x)$ at $x = 1$.

6 (8 pts). Sketch the graph of a function that clearly possesses all the given properties. Your graph needn't be drawn to scale. Label points and lines as necessary for clarity.

$$f(0) = 1,\quad f'(0) = 0,\quad f(-4) = 0,\quad f'(-4) = 1$$

$$\lim_{x\to -3^-} f(x) = 1,\quad \lim_{x\to -3^+} f(x) = 2,\quad \lim_{x\to 2} f(x) = \infty,$$

7 (13 pts). Use the definition of the derivative as a limit to find $r'(x)$ if $r(x) = \sqrt{2-3x}$.

1a. (Source: 2.4.Definition 2) For every positive number ε, there's a corresponding positive number δ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

1b. (Source: 2.4.16) To come up with proof, it helps to do some analysis first. Given $\varepsilon > 0$, we're looking for a number δ so that $0 < |x-2| < \delta$ will imply that $|(5x-13)-(-3)| < \varepsilon$. Start by simplifying:

$$|(5x-13)-(-3)| = |5x-10| = |5(x-2)| = |5||x-2| = 5|x-2|,$$

so to make $5|x-2| < \varepsilon$, just make sure that $|x-2| < \frac{1}{5}\varepsilon$.

Now we're ready to write a proof.

Proof: Suppose that $\varepsilon > 0$. Choose $\delta = \frac{1}{5}\varepsilon$. Then, whenever

$$|x-2| < \delta = \frac{1}{5}\varepsilon,$$

$$|(5x-13)-(-3)| = |5x-10| = 5|x-2| < 5 \cdot \frac{1}{5}\varepsilon = \varepsilon,$$

as desired.

2a. (Source: 2.3.26) As $x \to 2$, the function looks like "$\frac{1}{0} - \frac{2}{0}$," which, by itself, tells us nothing about the limit. (After all, a huge number minus a huge number could turn out to be anything.) Add the fractions:

$$\frac{1}{2-x} - \frac{2}{(2-x)x} = \frac{x}{(2-x)x} - \frac{2}{(2-x)x} = \frac{x-2}{(2-x)x} = \frac{x-2}{-(x-2)x} = \frac{1}{-x}.$$

Now let $x \to 2$, and the limit is $\lim\limits_{x\to 2} \frac{1}{-x} = -\frac{1}{2}$.

2b. (Source: 2.6.31,34) It's easiest to use the fact that the limit as $x \to \pm\infty$ of a rational function is the same as the limit of the ratio of its lead terms:

$$\lim_{x\to-\infty} \frac{4x^5+6x}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{4x^5}{2x^3} = \lim_{x\to-\infty} 2x^2 = +\infty.$$

If you didn't use this, then you could still factor out the lead term of the numerator and denominator:

$$\lim_{x\to-\infty} \frac{4x^5+6x}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{4x^5(1+\frac{3}{2x^4})}{2x^3(1-\frac{2}{x^2}+\frac{5}{2x^3})} = \lim_{x\to-\infty} \frac{2x^2(1+\frac{3}{2x^4})}{(1-\frac{2}{x^2}+\frac{5}{2x^3})}$$
$$= \lim_{x\to-\infty} 2x^2 \lim_{x\to-\infty} \frac{(1+\frac{3}{2x^4})}{(1-\frac{2}{x^2}+\frac{5}{2x^3})} = \infty \cdot 1 = \infty.$$

2c. (Source: 2.3.16) Factor and cancel:

$$\lim_{x\to-2} \frac{3x^2+2x-8}{x^2+5x+6} = \lim_{x\to-2} \frac{(3x-4)(x+2)}{(x+3)(x+2)} = \lim_{x\to-2} \frac{3x-4}{x+3} = -10.$$

2d. (Source: 2.2.41) As $x \to -3$, $\frac{3x^2+2x-8}{x^2+5x+6} = \frac{3x-4}{x+3}$ looks like "$\frac{-13}{0}$," indicating blow-up. Examine the signs. The numerator is near -13 and therefore must be negative. When $x > -3$ as in part d., $x+3 > 0$, so the fraction is $\frac{-}{+} = -$ and so $\lim_{x\to -3^+} \frac{3x-4}{x+3} = -\infty$.

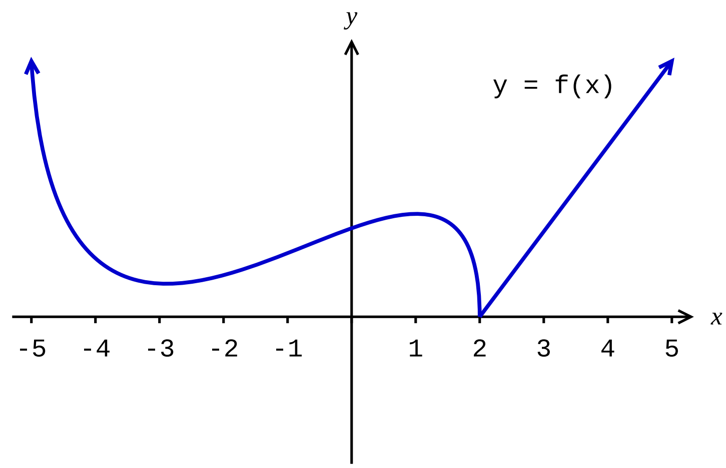
2e. (Source: 2.2.41) Continuing, when $x < -3$, $x+3 < 0$, so the fraction is $\frac{-}{-} = +$ and so $\lim_{x\to -3^+} \frac{3x-4}{x+3} = \infty$.

2f. (Source: 2.6.18) As in part b., $\lim_{x\to\infty} \frac{3x^2+2x-8}{x^2+5x+6} = \lim_{x\to\infty} \frac{3x^2}{x^2} = \lim_{x\to\infty} 3 = 3$.

3. (Source: 2.6.49,51) $y = \frac{3x^2+2x-8}{x^2+5x+6}$ can only blow up where it is undefined, and between 2 and -3, is only blows up at $x = -3$, so this is the only vertical asymptote. Since $y \to 3$ as $x \to \infty$, the horizontal asymptote is $y = 3$.

4. (Source: 2.8.3,5,7) Here's the graph of f and its derivative.

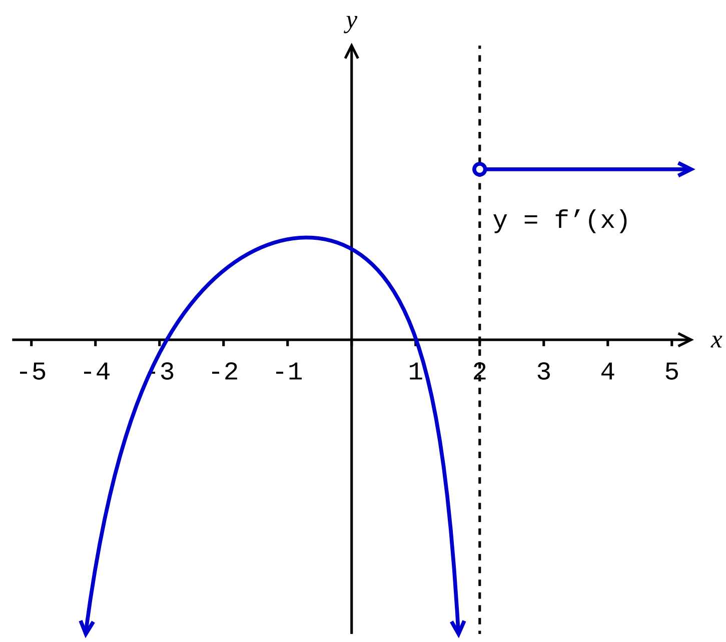
Note that when $y = f(x)$ has positive [negative] slope, $y = f'(x)$ has positive [negative] altitude. When the line tangent to $y = f(x)$ is horizontal, $y = f'(x)$ has a zero. $f' \to -\infty$ as $x \to 2^-$, since the tangent line is becoming vertical. To right right of 2, f' is a positive constant, since f has positive constant slope. $f'(2)$ does not exist because at $x = 2$, f has a corner (or cusp, or vertical tangent line, depending on your point of view).



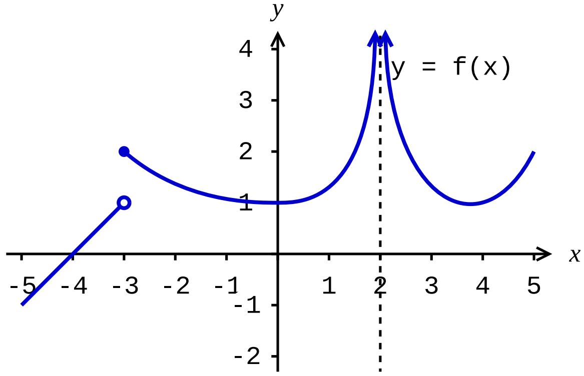
5. (Source: 3.1.4,6,13,16,46)

$$s(x) = 3x^5 - 2x^{-5} - 3x^{2/3} + e^x$$
$$s'(x) = 15x^4 + 10x^{-6} - 2x^{-1/3} + e^x$$
$$s''(x) = 60x^3 - 60x^{-7} + \tfrac{2}{3}x^{-4/3} + e^x$$

To find a line, we need a point on the line and its slope. In the case of a tangent line, the point is the point of tangency, in this case $(1, s(1)) = (1, e-2)$. (Note $e^1 = e$, not 0.) The derivative tells us the slope, or $s'(1) = 23+e$. In point-slope form, the line can be written
$y - e + 2 = (23+e)(x-1)$.



6. (Source: a: 2.2.15, 2.5.7, 2.6.6, b: 2.7.23) Here is one possible graph.



The curve must pass through $(4, 0)$ with slope 1 and through $(0, 1)$ with slope 0, i.e., with horizontal tangent. The problem doesn't say anything about $f(-3)$. I made my function right-continuous, but it's only important that your graph not violate the vertical line test. As x goes to 2 from either the right or the left, y must go to $+\infty$.

7. (Source: 2.7.35)

$$\begin{aligned}
r'(x) &= \lim_{h\to 0} \frac{r(x+h)-r(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2-3(x+h)}-\sqrt{2-3x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{2-3(x+h)}-\sqrt{2-3x}}{h}\right)\left(\frac{\sqrt{2-3(x+h)}+\sqrt{2-3x}}{\sqrt{2-3(x+h)}+\sqrt{2-3x}}\right) \\
&= \lim_{h\to 0} \frac{2-3(x+h)-(2-3x)}{h(\sqrt{2-3(x+h)}+\sqrt{2-3x})} = \lim_{h\to 0} \frac{-3h}{h(\sqrt{2-3(x+h)}+\sqrt{2-3x})} \\
&= \lim_{h\to 0} \frac{-3}{\sqrt{2-3(x+h)}+\sqrt{2-3x}} \\
&= \frac{-3}{2\sqrt{2-3x}}.
\end{aligned}$$