MATH 120–04 (Kunkle), Exam 1
100 pts, 75 minutes  Sept. 9, 2014  Page 1 of 1

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

You will not receive credit for using l’Hospital’s Rule (a technique learned later in a calculus course) on any problem on this exam.

When asked to find \( f'(x) \), you are expected to do so using the definition of derivative as a limit, and not the shortcut methods learned later in a calculus course.

2.4 1 (6 pts). State the precise definition of \( \lim_{x \to a} h(x) = P \)

2.4 2 (8 pts). Write an \( \varepsilon-\delta \) proof of \( \lim_{x \to 1} (5x - 3) = 2 \)

2.7, 2.8 3a (15 pts). Use the definition of derivative to find \( f'(a) \) if \( f(x) = \frac{2x}{4-x} \)

2.8 3b (3 pts). Suppose that the position (in meters) of an object moving along a number line is \( \frac{2x}{x} \) at time \( x \) (measured in seconds). Use your answer to 3a to find the object’s velocity at time 6 seconds. Don’t forget the units in your answer.

2.2, 2.5 (13 pts). On the grid provided, sketch the graph of a function \( f(x) \) that clearly possesses all of the following properties. Attach arrows \((\rightarrow)\) to the ends of your curve to show its direction any place that it leaves the grid.

2.7, 2.8 a. The domain of \( f(x) \) is \((-\infty, -1) \cup (-1, 6]\), that is, all \( x \leq 6 \) except \(-1\).

b. \( \lim_{x \to -1^-} f(x) = -\infty \)

c. \( \lim_{x \to -1^+} f(x) = \infty \)

d. \( \lim_{x \to \infty} f(x) = 2 \)

e. \( \lim_{x \to 0} f(x) \) does not exist.

f. \( \lim_{x \to 2} f(x) = -1 \)

g. \( f(x) \) is discontinuous at \( x = 2 \).

h. \( f(-5) = 1 \)

i. \( f'(-5) = 0 \)

j. \( \lim_{x \to -5} f'(x) = \infty \)

k. \( f(-3) = 4 \)

l. \( f'(-3) > 0 \)

2.3 5 (10 pts). Evaluate the limit. One-word answers are not acceptable for full credit, but your final answer to each part should be a number, \( \infty \), \(-\infty \), or DNE.

\[ \lim_{h \to 0} \frac{(1+h)^4 - 1}{h} \]

2.2, 2.3 6 (23 pts). Evaluate the limit. One-word answers are not acceptable for full credit, but your final answer to each part should be a number, \( \infty \), \(-\infty \), or DNE.

2.6 a. \( \lim_{x \to 3} \frac{\sqrt{3} - \sqrt{x}}{3 - x} \)

b. \( \lim_{x \to 2} \frac{x^3 - 4x}{2x^2 + 7x + 6} \)

c. \( \lim_{x \to \infty} \frac{x^3 - 4x}{2x^2 + 7x + 6} \)

d. \( \lim_{x \to \infty} \frac{x^3 - 4x}{2x^2 + 7x + 6} \)

2.2, 2.5 7 (22 pts). Evaluate the limit. One-word answers are not acceptable for full credit, but your final answer to each part should be a number, \( \infty \), \(-\infty \), or DNE.

a. \( \lim_{x \to -1} \frac{2x + 1}{(x + 1)^2} \)

b. \( \lim_{x \to 2} \frac{x + 1}{x - 2} \)

c. \( \lim_{x \to 7} \ln |x - 8| \)

d. \( \lim_{x \to 8} \ln |x - 8| \)
Note: "n.p." = no penalty. A small error worth less than one point.

1. For every $\varepsilon > 0$ there's a corresponding number $\delta > 0$ so that
   $0 < |x - 8| < \delta$ implies $|h(x) - P| < \varepsilon$. [Extra credit: Problem 1 was worth 7 pts]

   Alternate Form of this question: Replace 8 with $r$ and P with $Q$.

2. Suppose $\varepsilon > 0$. Choose $\delta = \varepsilon/5$. Then $0 < |x - 1| < \delta$ implies that
   $|5x - 3 - 2| = |5x - 5| = |5||x - 1| < 5\delta = 5\varepsilon/5 = \varepsilon$, as desired.
   Alt: Let $\varepsilon > 0$. Choose $\delta = \varepsilon/7$. Then $0 < |x - 1| < \delta$ implies that
   $|7x - 3 - 4| = |7x - 7| = |7||x - 1| < 7\delta = 7\varepsilon/7 = \varepsilon$, as desired.

3a. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2(4a + 8h - a^2 - 2ah) - \frac{8}{h}}{h(4a - h)(4a)}$

   $= \lim_{h \to 0} \frac{2(4a + 8h - a^2 - 2ah)}{h(4a - h)(4a)} = \lim_{h \to 0} \frac{2(4a + 8h - a^2 - 2ah)}{h(4a - h)(4a)}$

   Alt: Replace 4 by 6 $\Rightarrow$ Replace 8 by 12. ans: $f'(x) = \frac{12}{(6 - a)^2}$

3b. Velocity = $f'(6) = \frac{3}{(2)^2} = 2$ m/sec. Ans: $f'(8) = \frac{12}{(12)^2} = 3$ m/sec
4 (13 pts). On the grid provided, sketch the graph of a function \( f(x) \) that clearly possesses all of the following properties. Attach arrows (→) to the ends of your curve to show its direction anywhere that it leaves the grid.

a. The domain of \( f(x) \) is \((-∞, -1) \cup (-1, 6]\), that is, all \( x \leq 6 \) except \(-1\).

b. \( \lim_{{x \to -1^-}} f(x) = \infty \).

c. \( \lim_{{x \to -1^+}} f(x) = -\infty \).

d. \( \lim_{{x \to \infty}} f(x) = -2 \).

e. \( \lim_{{x \to 0}} f(x) \) does not exist.

f. \( \lim_{{x \to 2^0}} f(x) = 1 \).

g. \( f(x) \) is discontinuous at \( x = 2 \).

h. \( f(-5) = -1 \).

i. \( f'(-5) = 0 \).

j. \( \lim_{{x \to 6^-}} f'(x) = \infty \).

k. \( f(-3) = -4 \).

l. \( f'(-3) < 0 \). These parts were thrown out.

Alt:

a. The domain of \( f(x) \) is \((-∞, -1) \cup (-1, 6]\), that is, all \( x \leq 6 \) except \(-1\).

b. \( \lim_{{x \to -1^-}} f(x) = -\infty \).

c. \( \lim_{{x \to -1^+}} f(x) = \infty \).

d. \( \lim_{{x \to \infty}} f(x) = 2 \).

e. \( \lim_{{x \to 0}} f(x) \) does not exist.

f. \( \lim_{{x \to 2^0}} f(x) = -1 \).

g. \( f(x) \) is discontinuous at \( x = 2 \).

h. \( f(-5) = 1 \).

i. \( f'(-5) = 0 \).

j. \( \lim_{{x \to 6^-}} f'(x) = \infty \).

k. \( f(-3) = 4 \).

l. \( f'(-3) > 0 \). These parts were thrown out.
5. Use Pascal's triangle: \[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 4 \\
\end{array}
\]
\[
\lim_{h \to 0} \frac{(1+h)^4-1}{h} = \lim_{h \to 0} \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-1}{h} = \lim_{h \to 0} \frac{h(4x^3+4x^2h^2+4xh^3+h^4)}{h} = 4. \\
\]

6. a. \[
\lim_{x \to 3} \frac{(\sqrt{3} - \sqrt{x})}{(\sqrt{3} + \sqrt{x})} = \lim_{x \to 3} \frac{(3-x)}{x(3-x)(\sqrt{3} + \sqrt{x})} = \frac{1}{1(3+3)} = \frac{1}{6}. \\
\]

7. (a) Replace 3 throughout by 5.

b. \[
\lim_{x \to 2} \frac{x^3-4x}{2x^2+7x+6} = \lim_{x \to 2} \frac{x(x-2)(x+2)}{(x+2)(2x+3)} = \lim_{x \to 2} \frac{x(x-2)}{(2x+3)} = \frac{-2(2-2)}{2(2+3)} = -\frac{4}{10} = -\frac{2}{5}. \\
\]

5. d. As in c, \[
\lim_{x \to 0} \frac{\sin x}{x} = 1. \\
\]

Alt: d. \[
\text{c. and d. were worth } 9 \text{ pts together.} \\
\]

7. a. \[
\lim_{x \to 1} \frac{2x+1}{(x+1)^2} = \frac{-1}{0} \Rightarrow \text{Blow Up. Only possible limits are } \pm \infty. \\
\]

Check signs: \[
\frac{2x+1}{(x+1)^2} = \frac{-1}{0} \Rightarrow \text{Blow up. Conclude } \lim_{x \to 1} \frac{2x+1}{(x+1)^2} = -\infty. \\
\]

b. \[
\lim_{x \to 2} \frac{x+1}{x-2} = \frac{3}{0} \Rightarrow \text{Blow up. Sign? } \frac{x+1}{x-2} = \frac{3}{0}, \text{ depending on } x \to 2. \\
\]

Conclusion: \[
\lim_{x \to 2} \frac{x+1}{x-2} \text{ does not exist.} \\
\]

C. As \(x \to 1\), \(f(x) = x - 1\) is continuous. Consequently, \(\lim_{x \to 1} f(x) = f(1) = 0\).

D. As \(x \to 0\), \(|x| \to 0\), so \(\lim_{x \to 0} x = 0\). (Recall graph of \(x^n\).)