0.1: The Binomial Theorem and Pascal's Triangle.

The formulas
\[(x + y)^2 = x^2 + 2xy + y^2, \text{ and} \]
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.\]

are special instances of the Binomial Theorem, which says that the coefficients in the expansion
\[(x + y)^n = x^n + ?x^{n-1}y + ?x^{n-2}y^2 + \ldots + ?x^2y^{n-2} + ?xy^{n-1} + y^n\]

are found in Pascal’s Triangle:

\[
\begin{array}{cccc}
1 & & & \\
1 & 1 & & \\
1 & 2 & 1 & \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Details can be found in http://kunklet.people.cofc.edu/MATH111/pascal.pdf

0.1.re.e1.
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]
\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]

0.1.re.e2. Generate the next three rows of Pascal’s Triangle.

0.1.re.e3. Expand the following.

a. \((x + 3)^4\)  
   b. \((u + v)^5\)  
   c. \((u - v)^6\)  
   d. \((x^3 + y)^4\)  
   e. \((x - x^{-1})^5\)  
   f. \((\xi - 2)^6\)

Answers

0.1.re.e2. 5th row: 1 5 10 10 5 1. 6th row: 1 6 15 20 15 6 1. 7th row: 1 7 21 35 35 21 7 1.
0.1.re.e3a. \(x^4 + 12x^3 + 54x^2 + 108x + 81\)  
0.1.re.e3b. \(v^5 + 5v^4 + 10v^3 + 10v^2 + 5v + 1\)  
0.1.re.e3c. \(u^6 - 6u^5v + 15u^4v^2 - 20u^3v^3 + 15u^2v^4 - 6uv^5 + v^6\)  
0.1.re.e3d. \(x^{12} + 4x^9y + 6x^6y^2 + 4x^3y^3 + y^4\)  
0.1.re.e3e. \(x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} + x^{-5}\)  
0.1.re.e3f. \(\xi^6 - 12\xi^5 + 60\xi^4 - 160\xi^3 + 240\xi^2 - 192\xi + 64\)
Ap.D: Trigonometry

For a more complete review of trigonometry, see Appendix D of our text. The two basic functions in trigonometry are the sine and cosine, graphed here:

The other four trig functions are defined using sine and cosine:

\[
\begin{align*}
\tan x &= \frac{\sin x}{\cos x} & \cot x &= \frac{\cos x}{\sin x} \\
\sec x &= \frac{1}{\cos x} & \csc x &= \frac{1}{\sin x}
\end{align*}
\]

sin \( x \) and \( \cos x \) are defined for all real numbers \( x \), but \( \tan x \) and \( \sec x \) are undefined whenever \( \cos x = 0 \), and \( \cot x \) and \( \csc x \) are undefined whenever \( \sin x = 0 \).

By definition, \( \cos x \) and \( \sin x \) are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin) \( x \) radians counterclockwise from the positive horizontal axis.

Consequently, the ray through the origin \( x \) radians from the positive horizontal axis has slope \( \tan x \), and, when \( x \) is an acute angle, \( \cos x \) and \( \sin x \) are the legs of this right triangle with hypotenuse 1 and interior angle \( x \).
These basic trigonometric identities follow from the definitions of sine and cosine.

<table>
<thead>
<tr>
<th>PYTHAGOREAN IDENTITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin^2 x + \cos^2 x = 1 )</td>
</tr>
<tr>
<td>( \tan^2 x + 1 = \sec^2 x )</td>
</tr>
<tr>
<td>( 1 + \cot^2 x = \csc^2 x )</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>EVEN &amp; ODD IDENTITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(-x) = \cos x )</td>
</tr>
<tr>
<td>( \sin(-x) = -\sin x )</td>
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</tbody>
</table>

**Sketching the sine and cosine**

The sine and cosine are periodic functions having period \( 2\pi \), meaning

\[
\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x.
\]

To sketch one cycle of the sine, plot the vertical coordinates on the unit circle at the angles \( 0, \pi/2, \pi, 3\pi/2, \) and \( 2\pi \). Be careful to make these five points equally spaced horizontally and vertically. Then connect them with a smooth curve.

To sketch the cosine, start by plotting the horizontal coordinates on the unit circle at the same five angles.

**Ap.D.re.e1.** Sketch the graphs of the sine and cosine on the given interval. Label hashmarks so as to clearly indicate all points where \( y = -1, 0, 1 \) along your curve. (You can check your answers using Desmos.com.)

a. \([0, 2\pi]\)  
b. \([0, 3\pi]\)  
c. \([-\pi, \pi]\)
Known values of sine and cosine

We already know the values of sine and cosine at the four points where the unit circle intersects the $x$ and $y$ axes. By placing these two triangles:

![Triangles](image1)

around the unit circle like this:

![Unit Circle](image2)

we find the sines and cosines at 12 more points on the unit circle (and the infinitely many angles that reach those points).

Ap.D.re.e2. Find all angles whose cosine is $-\frac{1}{2}$.

We recognize $1/2$ as the short side of the 30-60-90 triangle, so for the cosine to be $-1/2$, the angle must be one of the two pictured at right. Find one angle that matches each drawing, for instance,

$$x = \pi - \pi/3 = 2\pi/3 \quad \text{and} \quad x = \pi + \pi/3 = 4\pi/3.$$  

Then add all multiples of $2\pi$ to describe all angles that fit the drawings:

$$x = 2\pi/3 + 2\pi n \quad \text{and} \quad x = 4\pi/3 + 2\pi n \quad \text{(where } n \text{ is any integer)}.$$


1. $\sin x = -1/\sqrt{2}$
2. $\cos x = 0$
3. $\tan x = -1$
4. $\sec x = -2$
5. $\csc x = 2/\sqrt{3}$
6. $\cot x = \sqrt{3}$
Solving trigonometric equations

When the variable of the equation appears inside a trig function, first solve for the function, and then solve for the variable.

Ap.D.re.e4. Solve for \( x \) in the equation \( 4 \sin^2 x - 8 \sin x + 3 = 0 \)

Solution: This is a quadratic equation in \( \sin x \) which can be solved by factoring:

\[
(2 \sin x - 3)(2 \sin x - 1) = 0
\]

which implies either \( \sin x = \frac{3}{2} \) or \( \sin x = \frac{1}{2} \). The first of these has no real solutions, so the solution set of the original equation is the same as the solution set of \( \sin x = \frac{1}{2} \), namely \( x = \pi/6 + 2\pi n \) or \( x = 5\pi/6 + 2\pi n \) (for any \( n \in \mathbb{Z} \)).

It sometimes help to use the Pythagorean identities to rewrite the equation entirely in terms of one trig function.

Ap.D.re.e5. Find all solutions \( x \) to the given equation.

a. \( 1 - \sin t - 2 \cos^2 t = 0 \) b. \( 3 \cos t - 2 \sin^2 t = 0 \) c. \( \sin^2 t + 3 \sin t + 2 - \cos^2 t = 0 \) d. \( 1 - \cos^2 t = 0 \) e. \( \cos^2 t - 3 = 0 \) f. \( 2 \sin^2 t - 2 \cos^2 t = 1 \)

The Law of Cosines

When we label the sides and any one angle of a triangle as shown in the figure, the Law of Cosines states that

\[
c^2 = a^2 + b^2 - 2ab \cos \theta.
\]

In case \( \theta \) is a right angle, the Law of Cosines reduces to the Pythagorean identity.

Ap.D.re.e6. Two ships leave the port of Charleston. One sails due east at a speed of 5 knots (nautical miles per hour) while the other sails in a direction 30° north of due east at 6 knots. What is the distance (in nautical miles) between the ships after two hours?

Answers

Ap.D.re.e3a. \( x = -\pi/4 + 2\pi n \) or \( x = -3\pi/4 + 2\pi n \)  Ap.D.re.e3b. \( x = \pi/2 + 2\pi n \) or \( x = -\pi/2 + 2\pi n \)

Ap.D.re.e3c. \( x = 3\pi/4 + 2\pi n \) or \( x = 11\pi/4 + 2\pi n \)  Ap.D.re.e3d. (same as in example Ap.D.re.e2)

Ap.D.re.e3e. \( x = \pi/3 + 2\pi n \) or \( x = 2\pi/3 + 2\pi n \)  Ap.D.re.e3f. \( x = \pi/3 + 2\pi n \) or \( x = 4\pi/3 + 2\pi n \)

Ap.D.re.e5a. \( \sin t = -1/2 \) or 1, \( t = \pi/2 + 2\pi n, 7\pi/2 + 2\pi n, 11\pi/6 + 2\pi n \)  Ap.D.re.e5b. \( \cos t = -2 \) (no sol’ns)

or 1/2, \( t = \pi/3 + 2\pi n, 5\pi/3 + 2\pi n \)  Ap.D.re.e5c. \( \sin t = -1/2 \) or 1, \( t = 3\pi/2 + 2\pi n, 7\pi/6 + 2\pi n, 11\pi/6 + 2\pi n \)

Ap.D.re.e5d. \( \cos t = \pm 1, t = n\pi \)  Ap.D.re.e5e. \( \cos t = \pm \sqrt{3} \) no real sol’ns  Ap.D.re.e5f. \( \sin t = \pm \sqrt{3}/2, t = \pi/3 + 2\pi n, 2\pi/3 + 2\pi n, 4\pi/3 + 2\pi n, 5\pi/3 + 2\pi n \)

Ap.D.re.e6. \( \sqrt{100 + 144 - 2 \cdot 10 \cdot 12 \cos \pi/6} = \sqrt{244 - 120\sqrt{3}} \)