The Binomial Theorem and Pascal's Triangle

There's an easy way to multiply out expressions of the form $(x+y)^n$, called **binomials**. The expansion must look something like

$$(x+y)^n = ?x^n + ?x^{n-1}y + ?x^{n-2}y^2 + \cdots + ?x^2y^{n-2} + ?xy^{n-1} + ?y^n. \tag{1}$$

Setting x or y equal zero tells us that the first and last coefficients must both be 1, but what about the others?

The **Binomial Theorem** tells us that the missing constants in (1), called the **binomial coefficients**, are found in the nth row of **Pascal's Triangle***:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

(Pascal's Triangle has infinitely many rows. We refer to the top row as its 0th row.) For instance, the 2nd row, "1 2 1," and the 3rd row, "1 3 3 1," tell us that

$$(x+y)^2 = x^2 + 2xy + y^2, \text{ and}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

To generate the next row, begin and end with a 1, and then add the two elements above to find the next entry. For example, since $1+3=4$, the next row will begin

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & & & & & 1 \end{array}$$

Finish the row with the sums $3+3$, and $3+1$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Consequently,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

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* Like many old objects in mathematics, this triangle isn't named for any of the various people who first discovered it, including mathematicians in China, Persia, and ancient India. See articles on Pascal's triangle at britannica.com, encyclopediaofmath.org, and wikipedia.org

To see why this works, consider the problem of expanding $(x+y)^3$. You could find $(x+y)^3$ by multiplying $(x+y)^2$ by $(x+y)$:

$$\begin{aligned}(x+y)^3 &= (x^2+2xy+y^2)(x+y)\\ &= x^3+2x^2y+xy^2\\ &\quad\;\; + x^2y \;\; + 2xy^2 + y^3\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

Similarly, you could find $(x+y)^4$ by multiplying $(x+y)^3$ by $(x+y)$:

$$\begin{aligned}(x+y)^4 &= (x^3+3x^2y+3xy^2+y^3)(x+y)\\ &= x^4+3x^3y+3x^2y^2+xy^3\\ &\quad\;\; + x^3y+3x^2y^2+3xy^3+y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4.\end{aligned}$$

Notice that each coefficient in the expansion of $(x+y)^3$ or of $(x+y)^4$, with the exception of the beginning and ending 1s, is the sum of the two neighboring coefficients in the row above. When we use one row of the Pascal's triangle to generate the next, we're just performing this process without all the symbols.

Example 1: Expand: $(x-y)^6$.

We'll need the 6th row of Pascal's Triangle. Starting where we left off,

$$\begin{array}{ccccccccccccc} & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Since $x-y$ is $x+(-y)$, Pascal's Triangle allows us to expand the binomial as powers of x and of $-y$:

$$(x+(-y))^6 = x^6+6x^5(-y)^1+15x^4(-y)^2+20x^3(-y)^3+15x^2(-y)^4+6x(-y)^5+(-y)^6$$

To get rid of the parentheses, observe that

$$\begin{array}{ll}(-y)^2=(-1)^2y^2=y^2, & (-y)^3=(-1)^3y^3=-y^3,\\ (-y)^4=y^4, & (-y)^5=-y^5,\\ (-y)^6=y^6, & (-y)^7=-y^7,\end{array}$$

etc., so the expansion simplifies to

$$(x-y)^6 = x^6-6x^5y^1+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$$

end Example 1

Exercises

1. Expand the binomial:

a. $(x+2)^3$	b. $(x-2)^3$	c. $(x+3)^4$	d. $(x-3)^5$
e. $(x+y)^7$	f. $(x-y)^7$	g. $\left(a-\dfrac{1}{a}\right)^8$	h. $(1-3x)^4$
i. $(2s-3)^5$	j. $(1+2x)^6$	k. $(1-v^2)^6$	l. $(u^2+1)^9$

Answers

1a. $x^3+6x^2+12x+8$ 1b. $x^3-6x^2+12x-8$ 1c. $x^4+12x^3+54x^2+108x+81$ 1d. $x^5-15x^4+90x^3-270x^2+405x-243$ 1e. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$ 1f. $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$ 1g. $a^8-8a^6+28a^4-56a^2+70-56a^{-2}+28a^{-4}-8a^{-6}+a^{-8}$ 1h. $1-12x+54x^2-108x^3+81x^4$ 1i. $32s^5-240s^4+720s^3-1080s^2+810s-243$ 1j. $1+12x+60x^2+160x^3+240x^4+192x^5+64x^6$ 1k. $1-6v^2+15v^4-20v^6+15v^8-6v^{10}+v^{12}$ 1l. $u^{18}+9u^{16}+36u^{14}+84u^{12}+126u^{10}+126u^8+84u^6+36u^4+9u^2+1$