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Complex Arithmetic

Math 111, College of Charleston

A **complex number** is an expression of the form

$$x + iy$$

where x and y are real numbers and i is the “imaginary” square root of -1. For example, $2+3i$ is a complex number. Just as we use the symbol $\mathbb{R}$ to stand for the set of real numbers, we use $\mathbb{C}$ to denote the set of all complex numbers. Any real number x is also a complex number, $x+0i$; in set notation, $\mathbb{R} \subset \mathbb{C}$.

Assume for this paragraph that

$$z = x + iy.$$

Then x is called the **real part** of z and y is called the **imaginary part** of z. This is written

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z).$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The **conjugate** of z is the complex number

$$\bar{z} = x - iy$$

and the **absolute value** of z is

$$|z| = \sqrt{x^2 + y^2}.$$

Note that when $y = 0$, this is the same as the absolute value formula for real numbers x. Note also that, since $(x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2$,

$$z\bar{z} = |z|^2.$$

You can add, subtract, multiply, and divide complex numbers using the usual rules of algebra, keeping in mind that $i^2 = -1$.

Example 1: *Write the sum in $x + iy$ form:*

$$(2+3i) + (4-i) = 6 + 2i$$

end Example 1

Example 2: *Write the sum in $x + iy$ form:*

$$(1+5i) + \overline{(2-3i)} = 1 + 5i + 2 + 3i = 3 + 8i$$

end Example 2

Example 3: *Write the number in $x + iy$ form:*

$$2(1-7i) - \frac{1}{2}(5+6i) = 2 - 14i - \frac{5}{2} - 3i = \frac{-1}{2} - 17i.$$

end Example 3

Example 4: *Find the product. Write your answer in $x + iy$ form:*

$$\begin{aligned}(4-7i)(2+3i) &= 8 - 14i + 12i - 21i^2 \\ &= 8 - (-1)21 - 2i \\ &= 29 - 2i\end{aligned}$$

end Example 4

Writing a quotient in $x + iy$ form requires the use of the conjugate, as the next example demonstrates.

Example 5: *Find the quotient. Write your answer in $x + iy$ form:*

$$\frac{4-7i}{2-3i} = \left(\frac{4-7i}{2-3i}\right)\left(\frac{2+3i}{2+3i}\right) = \frac{29-2i}{4+9} = \frac{29}{13} - \frac{2}{13}i$$

end Example 5

We sometimes have to use complex numbers in polynomials, as in the next example.

Example 6: *Expand the polynomial:*

$$\begin{aligned}(x-2+i)(x-2-i) &= x^2 - 2x + ix \\ &\qquad - 2x + 4 + 2i \\ &\qquad\quad - ix - 2i - i^2 \\ &= x^2 - 4x + 4 - i^2 = x^2 - 4x + 4 + 1 \\ &= x^2 - 4x + 5\end{aligned}$$

end Example 6

The easier way to solve such problems is to use the difference of squares:

Example 7: *Expand the polynomial:*

$$\begin{aligned}\big(x-3+4i\big)\big(x-3-4i\big) &= \big((x-3)+4i\big)\big((x-3)-4i\big) \\ &= (x-3)^2 - (4i)^2 \\ &= x^2 - 6x + 9 - 16i^2 = x^2 - 6x + 9 + 16 \\ &= x^2 - 6x + 25\end{aligned}$$

end Example 7

Exercises

1. Write in $x + iy$ form:

a. $3 + 2i + 2(1-i)$ b. $3(4-5i) - \overline{(2+4i)}$ c. $-2(2-i) + \frac{1}{3}(1+4i)$

d. $\frac{2}{3}(1+8i) + \frac{3}{2}(2-7i)$ e. $\frac{1}{5}(7-4i) - \frac{2}{3}\overline{(6-5i)}$ f. $(i+1)(i-1)$

g. $(2-3i)(2+3i)$ h. $(4-i)\overline{(5+2i)}$ i. $(3+\frac{1}{2}i)(\frac{1}{2}-\frac{1}{3}i)$

j. $(4+i) \div (1-8i)$ k. $\overline{(3-2i)} \div 2(1-i)$ l. $(1+2i) \div (1-2i)$

m. $(3+4i) \div (5+6i)$

2. Expand the polynomial:

a. $(x+2i)(x-2i)$ b. $(x-3i)(x+3i)$ c. $(x+i\sqrt{5})(x-i\sqrt{5})$

d. $(x-2+4i)(x-2-4i)$ e. $(x-3+i)(x-3-i)$ f. $(x+1-2i)(x+1+2i)$

g. $(x+\frac{1}{2}-i)(x+\frac{1}{2}+i)$ h. $(2x+1-i\sqrt{3})(2x+1+i\sqrt{3})$ i. $(x+3+i\sqrt{5})(x+3-i\sqrt{5})$

Answers

1a. 5 1b. $10-11i$ 1c. $\frac{-11}{3}+\frac{10}{3}i$ 1d. $\frac{11}{3}-\frac{31}{6}i$ 1e. $\frac{-13}{5}-\frac{52}{15}i$ 1f. -2 1g. 13 1h. $18-13i$ 1i. $\frac{5}{3}-\frac{3}{4}i$ 1j. $\frac{-4+12i}{65}$ 1k. $\frac{1}{4}+\frac{5}{4}i$ 1l. $\frac{-3}{4}+\frac{4}{5}i$ 1m. $\frac{39+2i}{61}$ 2a. x^2+4 2b. x^2+9 2c. x^2+5 2d. $x^2-4x+20$ 2e. $x^2-6x+10$ 2f. x^2+2x+5 2g. $x^2+x+\frac{5}{4}$ 2h. $4x^2+4x+4$ 2i. $x^2+6x+14$