MATH 111–03 (Kunkle), Exam 4

Name: ____________________________

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No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Unless a problem specifically refers to degrees, use radians in all questions and answers involving angles or trig functions. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

1. A radio antenna is mounted on the top of a tall building. From an observation sight on the ground 500 meters from the building, the angle of elevation to the top of the antenna is $81^\circ$, while the angle of elevation to the bottom of the antenna is $79^\circ$. How tall is the antenna? You should assume that the building is vertical and the ground is horizontal.

2a. Find $\sin x$, $\cos x$, and $\tan x$ for $x$ in the figure. Label your answers clearly.

2b. Find the length $y$ of the side in the figure.

3. Sketch one cycle of the graph of $y = 3 + \cos\left(\frac{\pi}{2}x - \frac{\pi}{4}\right)$. Draw the axes where you wish. Use and label the hashmarks so that your graph clearly shows the amplitude and the locations of all maximums, minimums, and zeros of the function that occur on your cycle.

4. Sketch one cycle of the graph of $y = -2 \csc x$. Draw the axes where you wish. Use and label the hashmarks so that your graph clearly shows the locations of all maximums, minimums, asymptotes, and intercepts that occur on your cycle.

5. Evaluate:
   a. $\sec(7\pi/3)$
   b. $\sin(11\pi/12)$
   c. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$
   d. $\tan(\sin^{-1}(1/4))$

6. Evaluate:
   a. $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$
   b. $\tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right)$

7. Suppose $\sin t = \frac{3}{4}$ and $\frac{\pi}{2} < t < \pi$. Find the following.
   a. $\cos\left(\frac{1}{2}t\right)$
   b. $\sin(2t)$

8a. Find all solutions $t$ to the equation $2\cos^2 t + 3\cos t = 2$.

8b. Find all solutions $t$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to the equation $2\tan^4 t + \sec^2 t - 2 = 0$. 
1. (Source: 4.11.14) In the picture, $u$ is the altitude of the bottom of the antenna, and $w$ is the altitude of the top. Then

$$\frac{u}{500} = \tan 79^\circ \quad \frac{w}{500} = \tan 81^\circ$$

$$u = 500 \tan 79^\circ \quad w = 500 \tan 81^\circ$$

The height of the antenna is $w - u = 500 \tan 81^\circ - 500 \tan 79^\circ$.

2a. (Source: 4.10.4) Use Pythagorus to find that the opposite leg has length $\sqrt{7}$. Then

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{7}}{4} \quad \cos x = \frac{\text{adj}}{\text{hyp}} = \frac{3}{4} \quad \tan x = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{7}}{3}$$

2b. (Source: 4.10.13) $y = \csc 13^\circ$, so $y = 4 \csc 13^\circ$.

3. (Source: 4.3.more.10) The function will go through one cycle when the angle inside the cosine goes from 0 to $2\pi$.

$$0 \leq \frac{\pi}{2} x - \frac{\pi}{4} \leq 2\pi$$

$$\frac{\pi}{4} \leq \frac{\pi}{2} x \leq 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$$

$$= \frac{1}{2} = \frac{2}{\pi} \cdot \frac{\pi}{4} \leq x \leq \frac{2}{\pi} \cdot \frac{9\pi}{4} = \frac{9}{2}$$

So, we’ll draw a cycle of the cosine starting at $x = \frac{1}{2}$ and ending at $x = \frac{9}{2}$. Raise the cycle three units, so that its maximum and minimum altitudes are 4 and 2:

![Graph of a cycle of cosine](centerline)

The five basic points to the graph: $(1/2, 4), (3/2, 3), (5/2, 2), (7/2, 3), \text{and} (9/2, 4)$.

4. (Source: 4.4.40) $\csc x$ is the reciprocal of $\sin x$, so the cosecant is positive when the sine is positive, negative when the sine is positive, $\pm 1$ when the sine is $\pm 1$, and has a vertical asymptote when the sine has a zero. On the left is a graph of $\sin x$ and $\csc x$, and, on the right, the graph of $-2 \csc x$. 

![Graph of sine and cosecant](centerline)
5a. (Source: 4.4.11, 4.5.6, 4.7.more.3g, 4.7.more.7b) 
\[ 7\pi/3 = 2\pi + \pi/3, \text{ so } \sec(7\pi/3) = \sec(\pi/3) = \frac{1}{\cos(\pi/3)} = \frac{1}{2} = 2. \]

5b. Use the addition formula for sine:
\[
\sin\left(\frac{11\pi}{12}\right) = \sin\left(\frac{9\pi}{12} + \frac{2\pi}{12}\right) = \sin\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)
= \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{3\pi}{4}\right) \sin\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.
\]

5c. \(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\) is the angle in \((-\pi/2, \pi/2)\) whose tangent is \(-1/\sqrt{3}\). That’s \(-\pi/6\).

5d. \(\sin^{-1}\left(\frac{1}{4}\right)\) is the acute angle whose sine is 1/4. Draw a triangle and label \(y\) and \(r\) 1 and 4, respectively. Find \(x = \sqrt{15}\) by Pythagorean. The tangent of this angle is \(y/x = 1/\sqrt{15}\).

6a. (Source: 4.7.more.5q) \(\frac{6\pi}{5}\) is in Quadrant III and its cosine is the \(x\)-coordinate of its terminal point. \(\cos^{-1}\left(\cos\left(\frac{6\pi}{5}\right)\right)\) is the angle \(\theta\) in \([0, \pi]\) for which \(\cos \theta = \cos\left(\frac{6\pi}{5}\right)\). The reference number of \(\frac{6\pi}{5}\) is \(\frac{\pi}{5}\), and so \(\theta\) must be \(\frac{4\pi}{5}\). (See the figure on the left below.)

6b. (Source: 4.7.more.6f) \(\tan\left(\frac{6\pi}{5}\right)\) is the slope the terminal side of \(\frac{6\pi}{5}\), and \(\tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right)\) is the angle \(\theta\) in \((-\pi/2, \pi/2)\) for which \(\tan \theta = \tan\left(\frac{6\pi}{5}\right)\). To have the same slope, the terminal sides of \(\theta\) and of \(\frac{6\pi}{5}\) must form a line, so \(\theta\) must be \(\frac{\pi}{5}\). (See the figure on right above.)

7. (Source: 4.5.45) \(\pi/2 < t < \pi\) means that the terminal side of \(t\) is in Quadrant II. Choose \(y = 3\) and and \(r = 4\). Find \(x\) by the
Pythagorean theorem, remembering that \( x < 0 \) in Quadrant II: 
\[ 4^2 = 3^2 + x^2 \implies x = -\sqrt{7} \]

7a. (Source: 4.5.29) By the half-angle identity for cosine, 
\[ \cos^2 \left( \frac{t}{2} \right) = \frac{1}{2} + \frac{1}{2} \cos t = \frac{1}{2} (1 - \frac{\sqrt{7}}{4}) \].
Since \( \frac{\pi}{4} < \frac{t}{2} < \frac{\pi}{2} \), \( \frac{t}{2} \) is in Quadrant I and its cosine is positive: 
\[ \cos \left( \frac{t}{2} \right) = \frac{1}{2} \left( 1 - \frac{\sqrt{7}}{4} \right) \].

7b. \( \sin(2t) = 2 \sin t \cos t = 2 \left( \frac{3}{4} \right) \left( -\frac{\sqrt{7}}{4} \right) = -\frac{3\sqrt{7}}{8} \)

8a. (Source: 4.8.23, 4.8.38) Get zero on one side and factor the other. 
\[ 0 = 2 \cos^2 t + 3 \cos t - 2 = (2 \cos t - 1)(\cos t + 2) \]. So, either \( \cos t = \frac{1}{2} \) or \( \cos t = -2 \). The latter has no real solutions. Solutions to the former are 
\[ t = \pm \frac{\pi}{3} + 2\pi n \] (See left figure below.)

8b. (Source: 4.8.38, 67) Use the Pythagorean identity for tan and sec to rewrite the equation entirely in terms of tan t:
\[ 0 = 2 \tan^4 t + (\tan^2 t + 1) - 2 = 2 \tan^4 t + \tan^2 t - 1 \]
\[ = (2 \tan^2 t - 1)(\tan^2 t + 1) = (\sqrt{2} \tan t - 1)(\sqrt{2} \tan t + 1)(\tan^2 t + 1) \]
So \( \tan t = \pm \frac{1}{\sqrt{2}} \) or \( \tan^2 t = -1 \) (which has no solutions). Solutions are 
\[ t = \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \]
and \( t = \tan^{-1} \left( -\frac{1}{\sqrt{2}} \right) \) (which equals \( -\tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \)). (These are not standard angles. See right figure above.)

I don’t recommend using sines and cosines in 8b, but if you did, here’s how it goes. Replace \( \tan t \) with \( \frac{\sin t}{\cos t} \) and \( \sec t \) with \( \frac{1}{\cos t} \), multiply the equation by \( \cos^4 t \) to clear the equation of fractions, and rewrite entirely in terms of sines. (Using cosines is trickier.)

\[
\begin{align*}
2 \left( \frac{\sin^4 t}{\cos^4 t} \right) + \frac{1}{\cos^2 t} - 2 &= 0 \\
2 \sin^4 t + \cos^2 t - 2 \cos^4 t &= 0 \\
2 \sin^4 t + \cos^2 t - 2(\cos^2 t)^2 &= 0 \\
2 \sin^4 t + 1 - \sin^2 t - 2(1 - \sin^2 t)^2 &= 0 \\
2 \sin^4 t + 1 - \sin^2 t - 2(1 - 2 \sin^2 t + \sin^4 t) &= 0 \\
2 \sin^4 t + 1 - \sin^2 t - 2 + 4 \sin^2 t - 2 \sin^4 t &= 0 \\
\sin^2 t &= 1/3 \\
sin t &= \pm \sqrt{1/3}
\end{align*}
\]

There are two \( t \)'s in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) that satisfy this equation: 
\( t = \sin^{-1} \left( \frac{\sqrt{1/3}}{3} \right) \) in \( (0, \frac{\pi}{2}) \) and 
\( t = \sin^{-1} \left( -\frac{\sqrt{1/3}}{3} \right) \) (which equals \( -\sin^{-1} \left( \frac{\sqrt{1/3}}{3} \right) \)) in \( (-\frac{\pi}{2}, 0) \).

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