MATH 111–03 (Kunkle), Exam 2 Name: ____________________

100 pts, 75 minutes Feb 28, 2017

No notes, books, electronic devices, or outside materials of any kind.

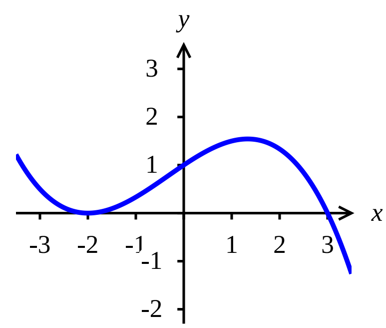
Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(15 pts). Find $f^{-1}(x)$ and its domain and range if $f(x) = \dfrac{2x}{5-3x}$. Label the domain and range clearly so I can tell which is which.

2(6 pts). Find two functions $y = f(x)$ and $y = g(x)$ defined implicitly by the equation $(y - x^2)(1 - x - y) = 0$. Graph each.

3(9 pts). Find a polynomial $p(x)$ of lowest possible degree whose graph is consistent with the graph shown at the right. Express $p(x)$ in factored form.



4(12 pts). A closed rectangular box is to be constructed with a square base. The material for the base costs \$3 per square foot, but the material for the top and vertical sides costs \$1 per square foot. If the volume of the box is to be 3 square cubic feet, express the total cost of the building materials as a function of the length of a side of the base.

5(12 pts). Find the difference quotient $\frac{f(x+h)-f(x)}{h}$ if $f(x) = x^3 - 4x + 1$. Cancel the factor h from top and bottom.

6(10 pts). Find the quotient and remainder in the division $(x^4 + 3x + 8) \div (x^3 - 2x^2)$.

7(14 pts). Show that $(x-3)(x+2)$ is a factor of $q(x) = x^4 + x^3 - 6x^2 - 14x - 12$. Then factor $q(x)$ completely and list all its zeros. Express any complex numbers in the form $a + ib$.

8a(8 pts). If a and b are integers, list all the possible rational zeros of the polynomial $r(x) = 2x^3 + ax^2 + bx + 4$.

8b(14 pts). Find all zeros of $s(x) = 2x^3 + 3x^2 - 10x + 4$ and factor $s(x)$ completely. Express any complex numbers in the form $a + ib$.

1.(Source: 2.8.41, more.1m) Solve for x:

$$y = f(x) = \frac{2x}{5-3x} \qquad 5y = 2x + 3xy$$
$$5y = x(2+3y)$$
$$(5-3x)y = 2x \qquad \frac{5y}{2+3y} = f^{-1}(y) = x$$
$$5y - 3xy = 2x$$

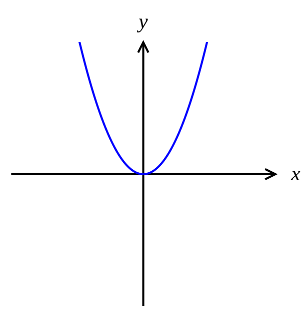
so $f^{-1}(x) = \frac{5x}{2+3x}$

$f^{-1}(x)$ is defined for all x except when $2 + 3x = 0$, so Domain f^{-1} is $(-\infty, -3/2) \cup (-3/2, \infty)$. Range of f^{-1} is the same as the Domain of f. Since f is undefined only when $5 - 3x = 0$, its domain is $(-\infty, 5/3) \cup (5/3, \infty)$.

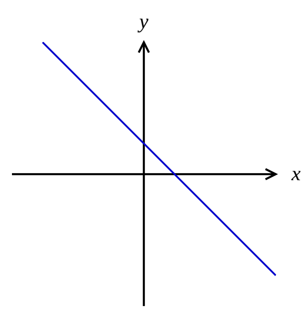
2.(Source: 2.7.7,10) For a product to equal zero, one of the factors must equal zero, so

$$(y - x^2)(1 - x - y) = 0$$
$$y - x^2 = 0 \quad \text{or} \quad 1 - x - y = 0$$
$$y = x^2 \quad \text{or} \quad 1 - x = y$$

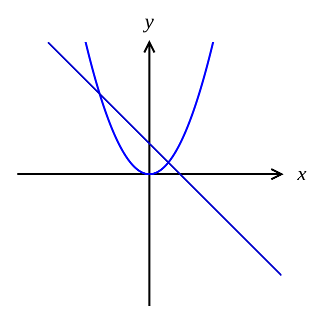
The graphs are a parabola and the line of slope -1 with y-intercept 1. (That means that the graph of the original equation is the union of these two.)



$y = x^2$

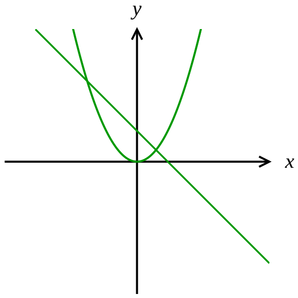


$y = 1 - x$

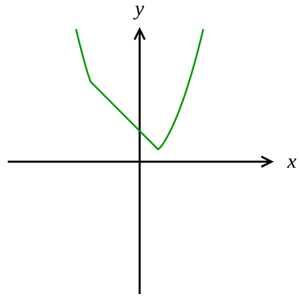


$(y - x^2)(1 - x - y) = 0$

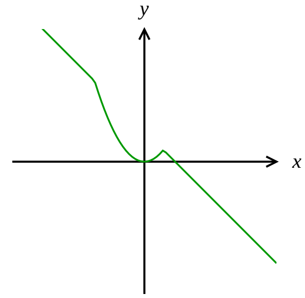
2. Alternate Solution: I don't recommend it, but you could multiply out the polynomial and then use the quadratic equation to find $f(x)$ and $g(x)$, i.e., the two possible values of y: $-y^2 + (1 - x + x^2)y - x^2(1-x) = 0 \Longrightarrow y = \frac{-(1-x+x^2) \pm \sqrt{(1-x+x^2)^2 - 4x^2(1-x)}}{-2}$, which, after some serious algebra, simplifies to $y = \frac{1-x+x^2 \pm |1-x-x^2|}{2}$ We already know what the two graphs together must look like, and from that we know the graphs of these two functions. We get the larger function using "+" and the smaller using "-" :



$(y-x^2)(1-x-y)=0$



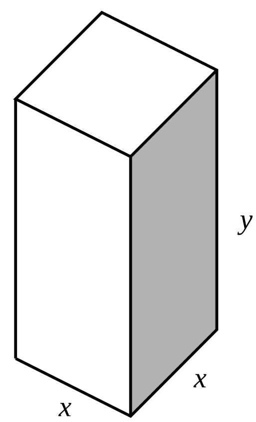
$y=\frac{1}{2}(1-x+x^2+|1-x-x^2|)$



$y=\frac{1}{2}(1-x+x^2-|1-x-x^2|)$

3.(Source: 3.1.46, 3.3.61) $p(x)$ has a zero of even multiplicity at $x=-2$ and a zero of multiplicity 1 at $x=3$, so, to keep the degree as small as possible, $p(x)=c(x+2)^2(x-3)$ for some constant c. Choose c so as to make the y-intercept equal 1: $1=c(0+2)^2(0-3)=-12c \Longrightarrow c=-1/12$. Therefore, $p(x)=-\frac{1}{12}(x+2)^2(x-3)$.

4.(Source: 2.9.38) Let x be the length of one side of the base and y the height of the box. Since cost = (cost per square foot)(area), the cost of materials for the box is \$3 times the area of the base plue \$1 times the area of the other sides:



$$C=3x^2+x^2+4xy=4x^2+4xy$$

The constraint is that volume of the box, x^2y, must equal 3. Solve for y in the constraint equation and substitute that in for y in the C-equation: $x^2y=3 \Longrightarrow y=3x^{-2} \Longrightarrow C=4x^2+4x\cdot 3x^{-2}=4x^2+12x^{-1}$

5.(Source: 2.10.17, more.1h)

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^3-4(x+h)+1-(x^3-4x+1)}{h}$$

To expand $(x+h)^3$, use the third row of Pascal's triangle: 1 3 3 1. Then distribute, collect up like terms, factor and cancel:

$$\frac{x^3+3x^2h+3xh^2+h^3-4x-4h+1-x^3+4x-1}{h}=\frac{3x^2h+3xh^2+h^3-4h}{h}$$

$$=\frac{h(3x^2+3xh+h^2-4)}{h}=3x^2+3xh+h^2-4$$

6.(Source: 3.2.7)

$$\begin{array}{r}
x+2 \\
x^3-2x^2 \overline{\big) \, x^4 \qquad\qquad +3x+8} \\
-(x^4-2x^3) \qquad\qquad \\
\hline
2x^3 \qquad +3x+8 \\
-(2x^3-4x^2) \qquad \\
\hline
4x^2+3x+8
\end{array}$$

The quotient is $x+2$ an the remainder is $4x^2+3x+8$.

7.(Source: 3.2.1-10, 23-32., 3.3.33) You could divide $q(x)$ by $(x-3)(x+2) = x^2-x-6$ using long division, but it's easier to divide twice synthetically:

$$\begin{array}{r|rrrrr}
3 & 1 & 1 & -6 & -14 & -12 \\
 & & 3 & 12 & 18 & -12 \\
\hline
-2 & 1 & 4 & 6 & 4 & 0 \\
 & & -2 & -4 & -4 & \\
\hline
 & 1 & 2 & 2 & 0 &
\end{array}$$

Consequently, $q(x) = (x-3)(x+2)(x^2+2x+2)$. You could find the zeros of the quadratic by the quadratic formula, or by completing the square as shown here:

$$0 = x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1 \Rightarrow (x+1)^2 = -1 \Rightarrow x+1 = \pm i \Rightarrow x = -1 \pm i$$

Therefore, $q(x) = (x-3)(x+2)(x+1-i)(x+1+i)$, and its zeros are $x = 3$, -2, $-1+i$, and $-1-i$.

8a.(Source: 3.4.4) The only possible rational zeros have numerators that divide 4 and denominators that divide 2:

$$\pm\left\{1, 2, 4, \frac{1}{2}, \frac{2}{2}, \frac{4}{2}, \frac{1}{4}, \frac{2}{4}, \frac{4}{4}\right\} = \pm\left\{1, 2, 4, \frac{1}{2}\right\}$$

8b.(Source: 3.4.more.1q) Of the possible rational roots in part a, $1/2$ is the only zero:

$$\begin{array}{r|rrrr}
1/2 & 2 & 3 & -10 & 4 \\
 & & 1 & 2 & -4 \\
\hline
 & 2 & 4 & -8 & 0
\end{array}$$

So, $s(x) = (x-1/2)(2x^2+4x-8) = 2(x-1/2)(x^2+2x-4)$ Now find the zeros of the quadratic either by completing the square or the quadratic formula, as shown here:
$x = \frac{1}{2}(-2 \pm \sqrt{2^2-4\cdot 1\cdot(-4)}) = \frac{1}{2}(-2 \pm \sqrt{20}) = \frac{1}{2}(-2 \pm 2\sqrt{5}) = -1 \pm \sqrt{5}$.

Therefore

$$\begin{aligned}
s(x) &= 2(x-1/2)(x-(-1+\sqrt{5}))(x-(-1-\sqrt{5})) \\
&= 2(x-1/2)(x+1-\sqrt{5})(x+1+\sqrt{5})
\end{aligned}$$

and its zeros are $x = 1/2$, $-1+\sqrt{5}$ and $-1-\sqrt{5}$.