1 (15 pts). Find $f^{-1}(x)$ and its domain and range if $f(x) = \frac{2x}{5-3x}$. Label the domain and range clearly so I can tell which is which.

2 (6 pts). Find two functions $y = f(x)$ and $y = g(x)$ defined implicitly by the equation $(y - x^2)(1 - x - y) = 0$. Graph each.

3 (9 pts). Find a polynomial $p(x)$ of lowest possible degree whose graph is consistent with the graph shown at the right. Express $p(x)$ in factored form.

4 (12 pts). A closed rectangular box is to be constructed with a square base. The material for the base costs $3 per square foot, but the material for the top and vertical sides costs $1 per square foot. If the volume of the box is to be 3 cubic feet, express the total cost of the building materials as a function of the length of a side of the base.

5 (12 pts). Find the difference quotient $\frac{f(x+h) - f(x)}{h}$ if $f(x) = x^3 - 4x + 1$. Cancel the factor $h$ from top and bottom.

6 (10 pts). Find the quotient and remainder in the division $(x^4 + 3x + 8) \div (x^3 - 2x^2)$.

7 (14 pts). Show that $(x - 3)(x + 2)$ is a factor of $q(x) = x^4 + x^3 - 6x^2 - 14x - 12$. Then factor $q(x)$ completely and list all its zeros. Express any complex numbers in the form $a + ib$.

8a (8 pts). If $a$ and $b$ are integers, list all the possible rational zeros of the polynomial $r(x) = 2x^3 + ax^2 + bx + 4$.

8b (14 pts). Find all zeros of $s(x) = 2x^3 + 3x^2 - 10x + 4$ and factor $s(x)$ completely. Express any complex numbers in the form $a + ib$. 
1. (Source: 2.8.41, more.1m) Solve for $x$:

$$y = f(x) = \frac{2x}{5-3x} \quad 5y = 2x + 3xy$$

$$(5-3x)y = 2x \quad 5y - 3xy = 2x$$

$$5y = 2 + 3y = f^{-1}(y) = x$$

so $f^{-1}(x) = \frac{5x}{2+3x}$

$f^{-1}(x)$ is defined for all $x$ except when $2 + 3x = 0$, so Domain $f^{-1}$ is $(-\infty, -3/2) \cup (-3/2, \infty)$. Range of $f^{-1}$ is the same as the Domain of $f$. Since $f$ is undefined only when $5 - 3x = 0$, its domain is $(-\infty, 5/3) \cup (5/3, \infty)$.

2. (Source: 2.7.7,10) For a product to equal zero, one of the factors must equal zero, so

$$(y - x^2)(1 - x - y) = 0$$

$$y - x^2 = 0 \quad \text{or} \quad 1 - x - y = 0$$

$$y = x^2 \quad \text{or} \quad 1 - x = y$$

The graphs are a parabola and the line of slope $-1$ with $y$-intercept 1. (That means that the graph of the original equation is the union of these two.)

2. Alternate Solution: I don’t recommend it, but you could multiply out the polynomial and then use the quadratic equation to find $f(x)$ and $g(x)$, i.e., the two possible values of $y$:

$$-y^2 + (1-x+x^2)y - x^2(1-x) = 0 \implies y = \frac{-(1-x+x^2) \pm \sqrt{(1-x+x^2)^2 - 4x^2(1-x)}}{-2}$$

which, after some serious algebra, simplifies to $y = \frac{1-x+x^2 \pm [1-x-x^2]}{2}$. We already know what the two graphs together must look like, and from that we know the graphs of these two functions. We get the larger function using “+” and the smaller using “-“.
3. (Source: 3.1.46, 3.3.61) $p(x)$ has a zero of even multiplicity at $x = -2$ and a zero of multiplicity 1 at $x = 3$, so, to keep the degree as small as possible, $p(x) = c(x + 2)^2(x - 3)$ for some constant $c$. Choose $c$ so as to make the $y$-intercept equal 1: $1 = c(0 + 2)^2(0 - 3) = -12c \implies c = -1/12$. Therefore, $p(x) = -\frac{1}{12}(x + 2)^2(x - 3)$.

4. (Source: 2.9.38) Let $x$ be the length of one side of the base and $y$ the height of the box. Since cost = (cost per square foot)(area), the cost of materials for the box is $3$ times the area of the base plus $1$ times the area of the other sides:

$$C = 3x^2 + x^2 + 4xy = 4x^2 + 4xy$$

The constraint is that volume of the box, $x^2y$, must equal 3. Solve for $y$ in the constraint equation and substitute that in for $y$ in the $C$-equation:

$$x^2y = 3 \implies y = \frac{3}{x} \implies C = 4x^2 + 4x \cdot \frac{3}{x} = 4x^2 + 12x^{-1}$$

5. (Source: 2.10.17, more.1h)

$$\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^3 - 4(x + h) + 1 - (x^3 - 4x + 1)}{h}$$

To expand $(x + h)^3$, use the third row of Pascal’s triangle: 1 3 3 1. Then distribute, collect up like terms, factor and cancel:

$$\frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h + 1 - x^3 + 4x - 1}{h} = \frac{3x^2h + 3xh^2 + h^3 - 4h}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2 - 4)}{h} = 3x^2 + 3xh + h^2 - 4$$
6. (Source: 3.2.7)

\[
x^3 - 2x^2 \frac{x^4}{x^4 - 2x^3} + 3x + 8
- \frac{2x^3}{2x^3 - 4x^2} + 3x + 8
\]

\[
\frac{4x^2 + 3x + 8}{4x^2 - 3x + 8}
\]

The quotient is \(x + 2\) and the remainder is \(4x^2 + 3x + 8\).

7. (Source: 3.2.1-10, 23-32., 3.3.33) You could divide \(q(x)\) by \((x - 3)(x + 2) = x^2 - x - 6\) using long division, but it’s easier to divide twice synthetically:

\[
\begin{array}{c|cccc}
 3 & 1 & 1 & -6 & -14 & -12 \\
  & 3 & 12 & 18 & 12 \\
-2 & 1 & 4 & 6 & 4 & 0 \\
  & -2 & -4 & -4 & \\
\hline
1 & 2 & 2 & 0 & \\
\end{array}
\]

Consequently, \(q(x) = (x - 3)(x + 2)(x^2 + 2x + 2)\). You could find the zeros of the quadratic by the quadratic formula, or by completing the square as shown here:

\[
0 = x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x + 1)^2 + 1 \Rightarrow (x + 1)^2 = -1 \Rightarrow x + 1 = \pm i \Rightarrow x = -1 \pm i
\]

Therefore, \(q(x) = (x - 3)(x + 2)(x + 1 - i)(x + 1 + i)\), and its zeros are \(x = 3, -2, -1 + i, \) and \(-1 - i\).

8a. (Source: 3.4.4) The only possible rational zeros have numerators that divide 4 and denominators that divide 2:

\[
\pm \left\{ 1, 2, \frac{1}{2}, 4, \frac{1}{4} \right\} = \pm \left\{ 1, 2, \frac{1}{2}, 4, \frac{1}{4} \right\} = \pm \left\{ 1, 2, \frac{1}{2} \right\}
\]

8b. (Source: 3.4.more.1q) Of the possible rational roots in part a, \(1/2\) is the only zero:

\[
\begin{array}{c|cccc}
\frac{1}{2} & 2 & 3 & -10 & 4 \\
  & 1 & 2 & -4 & \\
\hline
2 & 4 & -8 & 0 & \\
\end{array}
\]

So, \(s(x) = (x - 1/2)(2x^2 + 4x - 8) = 2(x - 1/2)(x^2 + 2x - 4)\) Now find the zeros of the quadratic either by completing the square or the quadratic formula, as shown here:

\[
x = \frac{1}{2}(-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-4)}) = \frac{1}{2}(-2 \pm \sqrt{20}) = \frac{1}{2}(-2 \pm 2\sqrt{5}) = -1 \pm \sqrt{5}.
\]

Therefore

\[
s(x) = 2(x - 1/2)(x - (1 + \sqrt{5}))(x - (1 - \sqrt{5})) = 2(x - 1/2)(x + 1 - \sqrt{5})(x + 1 + \sqrt{5})
\]

and its zeros are \(x = 1/2, -1 + \sqrt{5}\) and \(-1 - \sqrt{5}\).