Arithmetic with fractions

To add fractions with the same denominator, add the numerators:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \tag{1}$$

To multiply fractions, multiply the numerators and denominators:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \tag{2}$$

To divide fractions, invert and multiply:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \tag{3}$$

Division by zero is not allowed:

$$\frac{a}{0} \text{ does not exist} \tag{4}$$

(because $\frac{a}{0} = ?$ would mean $a = 0 \cdot ?$, which has no solution if $a \neq 0$ and infinitely many solutions if $a = 0$.) As a consequence of (2),

$$\frac{ac}{bc} = \frac{a}{b} \tag{5}$$

(because $\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1$.) We use this either to simplify a fraction by canceling common factors, e.g.,

$$\frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3}$$

or to obtain common denominators for adding fractions, e.g.,

$$\frac{4}{3} + \frac{3}{5} = \frac{5 \cdot 4}{5 \cdot 3} + \frac{3 \cdot 3}{5 \cdot 3} = \frac{20}{15} + \frac{9}{15} = \frac{29}{15}.$$

There's nothing inherently wrong with an improper fraction, but to write an improper fraction as a mixed number, we use long division. For example,

$$17 \div 5 = 3 \text{ with remainder } 2$$

means that

$$\frac{17}{5} = 3 + \frac{2}{5}.$$

1. Find the sum or difference.

a. $\frac{2}{5} + \frac{4}{5}$	b. $\frac{8}{13} - \frac{5}{13}$	c. $\frac{2}{9} + \frac{8}{9}$	d. $\frac{2}{8} - \frac{3}{8}$	e. $\frac{7}{6} + \frac{5}{6}$	f. $\frac{4}{3} - \frac{7}{3}$
g. $\frac{8}{14} - \frac{5}{14}$	h. $\frac{2}{5} + \frac{8}{5}$	i. $\frac{2}{20} - \frac{3}{20}$	j. $\frac{7}{11} + \frac{5}{11}$	k. $\frac{4}{15} - \frac{7}{15}$	l. $\frac{2}{23} + \frac{4}{23}$
m. $\frac{5}{3} + \frac{11}{3}$	n. $\frac{8}{4} - \frac{3}{4}$	o. $\frac{5}{28} + \frac{9}{28}$	p. $\frac{2}{7} - \frac{9}{7}$	q. $\frac{7}{5} + \frac{9}{5}$	r. $\frac{21}{9} - \frac{5}{9}$

2. Find the sum or difference.

a. $\frac{2}{3}+\frac{4}{5}$	b. $\frac{3}{4}-\frac{1}{5}$	c. $\frac{13}{3}+\frac{1}{8}$	d. $\frac{2}{11}+\frac{6}{7}$	e. $\frac{4}{21}-\frac{3}{11}$	f. $\frac{4}{5}+\frac{5}{4}$
g. $\frac{5}{14}-\frac{1}{15}$	h. $\frac{6}{52}+\frac{4}{3}$	i. $\frac{5}{12}-\frac{3}{5}$	j. $\frac{2}{3}+\frac{4}{7}$	k. $\frac{7}{10}-\frac{4}{9}$	l. $\frac{1}{30}-\frac{11}{7}$
m. $\frac{8}{3}+\frac{2}{13}$	n. $\frac{4}{9}-\frac{5}{7}$	o. $\frac{2}{3}+\frac{8}{5}$	p. $\frac{14}{5}-\frac{8}{9}$	q. $\frac{10}{3}+\frac{3}{10}$	r. $\frac{12}{11}-\frac{2}{3}$
s. $\frac{8}{15}+\frac{1}{6}$	t. $\frac{1}{2}-\frac{5}{11}$	u. $\frac{2}{11}+\frac{3}{5}$	v. $\frac{8}{15}+\frac{3}{2}$	w. $\frac{4}{15}-\frac{3}{4}$	x. $\frac{12}{5}-\frac{5}{11}$

3. Find the sum or difference. Hint: always use the *least* common demonimator.

a. $\frac{3}{4}-\frac{3}{10}$	b. $\frac{2}{9}+\frac{4}{21}$	c. $\frac{1}{22}+\frac{3}{8}$	d. $\frac{3}{10}-\frac{2}{15}$	e. $\frac{7}{8}+\frac{17}{12}$	f. $\frac{5}{18}-\frac{5}{24}$
g. $\frac{4}{21}-\frac{11}{14}$	h. $\frac{3}{12}+\frac{11}{28}$	i. $\frac{13}{12}-\frac{5}{9}$	j. $\frac{2}{14}+\frac{3}{8}$	k. $\frac{3}{13}+\frac{1}{52}$	l. $\frac{5}{9}-\frac{4}{21}$
m. $\frac{6}{10}-\frac{7}{15}$	n. $\frac{2}{5}+\frac{4}{25}$	o. $\frac{8}{3}+\frac{7}{9}$	p. $\frac{5}{8}-\frac{5}{24}$	q. $\frac{7}{11}+\frac{6}{55}$	r. $\frac{5}{9}+\frac{2}{15}$
s. $\frac{25}{21}-\frac{17}{15}$	t. $\frac{25}{24}+\frac{17}{20}$	u. $\frac{18}{15}-\frac{40}{35}$	v. $\frac{13}{6}+\frac{15}{8}$	w. $\frac{25}{12}-\frac{9}{8}$	x. $\frac{31}{10}+\frac{21}{16}$

4. Find the sum or difference. Hint: $n=\frac{n}{1}$

a. $\frac{2}{3}-4$	b. $\frac{4}{9}+2$	c. $\frac{13}{4}-2$	d. $\frac{11}{14}+3$	e. $\frac{7}{4}-5$	f. $\frac{8}{13}+2$
g. $10-\frac{5}{7}$	h. $5+\frac{4}{5}$	i. $4-\frac{1}{3}$	j. $3+\frac{2}{13}$	k. $2-\frac{5}{4}$	l. $-3+\frac{4}{9}$

5. Find the product. Hint: cancel common factors before multiplying.

a. $\frac{2}{3}\times\frac{0}{4}$	b. $\frac{5}{7}\times\frac{4}{3}$	c. $\frac{5}{0}\times\frac{11}{6}$	d. $\frac{2}{15}\times\frac{3}{5}$	e. $\frac{1}{3}\times\frac{25}{7}$	f. $\frac{-8}{5}\times\frac{2}{11}$
g. $\frac{8}{15}\times\frac{3}{10}$	h. $\frac{3}{-2}\times\frac{5}{6}$	i. $\frac{-3}{2}\times\frac{-7}{5}$	j. $\frac{-9}{-4}\times\frac{6}{6}$	k. $\frac{5}{4}\times\frac{6}{7}$	l. $\frac{14}{5}\times\frac{45}{21}$
m. $\frac{99}{25}\times\frac{5}{3}$	n. $\frac{8}{0}\times\frac{21}{24}$	o. $\frac{26}{15}\times\frac{35}{12}$	p. $\frac{5}{4}\times\frac{0}{3}$	q. $\frac{5}{13}\times\frac{3}{10}$	r. $\frac{8}{17}\times\frac{3}{16}$
s. $\frac{9}{2}\times\frac{7}{2}$	t. $\frac{4}{5}\times\frac{7}{5}$	u. $\frac{1}{3}\times\frac{4}{3}$	v. $\frac{2}{9}\times\frac{-7}{9}$	w. $\frac{4}{11}\times\frac{3}{11}$	x. $\frac{4}{-12}\times\frac{3}{12}$

6. Find the quotient. Remember that division by zero is not allowed.

a. $\frac{2}{5}\div\frac{11}{2}$	b. $\frac{8}{5}\div\frac{12}{35}$	c. $\frac{1}{2}\div\frac{-5}{7}$	d. $\frac{3}{14}\div\frac{12}{7}$	e. $\frac{1}{8}\div\frac{1}{9}$	f. $\frac{0}{3}\div\frac{1}{2}$
g. $\frac{8}{11}\div\frac{6}{5}$	h. $\frac{3}{-14}\div\frac{2}{-21}$	i. $\frac{3}{0}\div\frac{5}{1}$	j. $\frac{3}{7}\div\frac{9}{2}$	k. $\frac{4}{0}\div\frac{4}{9}$	l. $\frac{14}{5}\div\frac{7}{13}$
m. $\frac{22}{3}\div\frac{11}{15}$	n. $\frac{8}{3}\div\frac{21}{0}$	o. $\frac{27}{14}\div\frac{12}{49}$	p. $\frac{5}{18}\div\frac{0}{3}$	q. $\frac{3}{5}\div\frac{5}{-9}$	r. $\frac{8}{11}\div\frac{3}{7}$
s. $2\div\frac{1}{3}$	t. $4\div\frac{3}{4}$	u. $3\div\frac{2}{7}$	v. $12\div\frac{7}{3}$	w. $7\div\frac{1}{3}$	x. $8\div\frac{-1}{3}$
y. $\frac{1}{3}\div 4$	z. $\frac{2}{5}\div 2$	27. $\frac{5}{3}\div 3$	28. $\frac{2}{7}\div 8$	29. $\frac{-2}{7}\div 3$	30. $\frac{5}{3}\div 8$

Answers

1a. $\frac{6}{5}$ 1b. $\frac{3}{13}$ 1c. $\frac{10}{9}$ 1d. $-\frac{1}{8}$ 1e. 2 1f. -1 1g. $\frac{3}{14}$ 1h. 2 1i. $-\frac{1}{20}$ 1j. $\frac{12}{11}$ 1k. $-\frac{3}{15}$ 1l. $\frac{6}{23}$ 1m. $\frac{16}{3}$ 1n. $\frac{5}{4}$ 1o. $\frac{1}{2}$ 1p. -1 1q. $\frac{16}{5}$ 1r. $\frac{16}{9}$ 2a. $\frac{22}{15}$ 2b. $\frac{11}{20}$ 2c. $\frac{107}{24}$ 2d. $\frac{80}{77}$ 2e. $-\frac{19}{231}$ 2f. $\frac{41}{20}$ 2g. $\frac{61}{210}$ 2h. $\frac{113}{78}$ 2i. $-\frac{11}{60}$ 2j. $\frac{26}{21}$ 2k. $\frac{23}{90}$ 2l. $-\frac{323}{210}$ 2m. $\frac{110}{39}$ 2n. $-\frac{17}{63}$ 2o. $\frac{34}{15}$ 2p. $\frac{86}{45}$ 2q. $\frac{109}{30}$ 2r. $\frac{14}{33}$ 2s. $\frac{7}{10}$ 2t. $\frac{1}{22}$ 2u. $\frac{43}{55}$ 2v. $\frac{61}{30}$ 2w. $-\frac{29}{60}$ 2x. $\frac{107}{55}$ 3a. $\frac{9}{20}$ 3b. $\frac{26}{63}$ 3c. $\frac{37}{88}$ 3d. $\frac{1}{6}$ 3e. $\frac{55}{24}$ 3f. $\frac{5}{72}$ 3g. $-\frac{25}{42}$ 3h. $\frac{9}{14}$ 3i. $\frac{19}{36}$ 3j. $\frac{29}{56}$ 3k. $\frac{1}{4}$ 3l. $\frac{23}{63}$ 3m. $\frac{2}{15}$ 3n. $\frac{14}{25}$ 3o. $\frac{31}{9}$ 3p. $\frac{10}{24}$ 3q. $\frac{41}{55}$ 3r. $\frac{31}{45}$ 3s. $\frac{2}{35}$ 3t. $\frac{23}{120}$ 3u. $\frac{2}{35}$ 3v. $\frac{107}{24}$ 3w. $\frac{23}{24}$ 3x. $\frac{353}{80}$ 4a. $-\frac{10}{3}$ 4b. $\frac{22}{9}$ 4c. $\frac{5}{4}$ 4d. $\frac{53}{14}$ 4e. $-\frac{13}{4}$ 4f. $\frac{34}{13}$ 4g. $\frac{65}{7}$ 4h. $\frac{29}{5}$ 4i. $\frac{11}{3}$ 4j. $\frac{41}{13}$ 4k. $\frac{3}{4}$ 4l. $-\frac{23}{9}$ 5a. 0 5b. $\frac{20}{21}$ 5c. DNE 5d. $\frac{2}{25}$ 5e. $\frac{25}{21}$ 5f. $-\frac{16}{55}$ 5g. $\frac{4}{25}$ 5h. $-\frac{5}{4}$ 5i. $\frac{21}{10}$ 5j. $\frac{9}{4}$ 5k. $\frac{15}{14}$ 5l. 6 5m. $\frac{33}{5}$ 5n. DNE 5o. $\frac{91}{18}$ 5p. 0 5q. $\frac{3}{26}$ 5r. $\frac{3}{34}$ 5s. $\frac{63}{4}$ 5t. $\frac{28}{25}$ 5u. $\frac{4}{9}$ 5v. $-\frac{1}{4}81$ 5w. $\frac{12}{121}$ 5x. $-\frac{1}{12}$ 6a. $\frac{4}{55}$ 6b. $\frac{14}{3}$ 6c. $-\frac{7}{10}$ 6d. $\frac{1}{8}$ 6e. $\frac{9}{8}$ 6f. 0 6g. $\frac{520}{121}$ 6h. $\frac{9}{4}$ 6i. DNE 6j. $\frac{2}{21}$ 6k. DNE 6l. $\frac{26}{5}$ 6m. 10 6n. DNE 6o. $\frac{63}{8}$ 6p. DNE 6q. $-\frac{27}{25}$ 6r. $\frac{56}{33}$ 6s. 6 6t. $\frac{16}{3}$ 6u. $\frac{21}{2}$ 6v. $\frac{36}{7}$ 6w. 21 6x. -24 6y. $\frac{1}{12}$ 6z. $\frac{1}{5}$ 627. $\frac{5}{9}$ 628. $\frac{2}{56}$ 629. $-\frac{2}{21}$ 630. $\frac{5}{24}$

Rationalizing the denominator

It's sometimes handy to rewrite a fraction so that no radicals appear in the denominator. To do this, we multiply the fraction by 1 in an appropriate form, thus changing the appearance of the fraction but not its actual value.

Case 1: Rationalizing $\dfrac{a}{b\sqrt{c}}$.

In this case we can rationalize the denominator by multiplying by $1 = \frac{\sqrt{c}}{\sqrt{c}}$. For example,

$$\frac{14}{3\sqrt{7}} = \frac{14}{3\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{14\sqrt{7}}{3(\sqrt{7})^2} = \frac{14\sqrt{7}}{3 \cdot 7} = \frac{14\sqrt{7}}{21} = \frac{2\sqrt{7}}{3}.$$

Case 2: Rationalizing $\dfrac{a}{b \pm c\sqrt{d}}$.

When the last operation in the denominator is addition or subtraction, we rationalize the denominator by multiplying by $1 = \frac{b \mp c\sqrt{d}}{b \mp c\sqrt{d}}$ and using

$$(x-y)(x+y) = x^2 - y^2.$$

($b \mp c\sqrt{d}$ is called the **conjugate** of $b \pm c\sqrt{d}$.) For instance,

$$\frac{3}{5-2\sqrt{7}} = \frac{3}{5-2\sqrt{7}} \cdot \frac{5+2\sqrt{7}}{5+2\sqrt{7}} = \frac{3(5+2\sqrt{7})}{5^2-(2\sqrt{7})^2} = \frac{3(5+2\sqrt{7})}{25-4\cdot 7} = \frac{3(5+2\sqrt{7})}{-3} = -(5+2\sqrt{7})$$

Tip: look for common factors in the numerator and denominator and for perfect squares under the radical, e.g., $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$.

7. Rationalize the denominator.

a. $\frac{2}{\sqrt{5}}$	b. $\frac{4}{\sqrt{7}}$	c. $\frac{7}{\sqrt{5}}$	d. $\frac{9}{\sqrt{11}}$	e. $\frac{11}{2\sqrt{6}}$	f. $\frac{7}{3\sqrt{10}}$	g. $\frac{4}{5\sqrt{3}}$	h. $\frac{6}{7\sqrt{5}}$
i. $\frac{2}{\sqrt{6}}$	j. $\frac{4}{\sqrt{12}}$	k. $\frac{3}{\sqrt{15}}$	l. $\frac{9}{\sqrt{21}}$	m. $\frac{3}{2\sqrt{6}}$	n. $\frac{15}{2\sqrt{6}}$	o. $\frac{9}{5\sqrt{3}}$	p. $\frac{6}{5\sqrt{14}}$
q. $\frac{2}{\sqrt{12}}$	r. $\frac{4}{\sqrt{20}}$	s. $\frac{3}{\sqrt{8}}$	t. $\frac{9}{\sqrt{16}}$	u. $\frac{9}{2\sqrt{18}}$	v. $\frac{35}{2\sqrt{75}}$	w. $\frac{18}{3\sqrt{40}}$	x. $\frac{4}{3\sqrt{24}}$

8. Rationalize the denominator.

a. $\frac{3}{1+\sqrt{5}}$	b. $\frac{4}{2-\sqrt{7}}$	c. $\frac{3}{2+\sqrt{3}}$	d. $\frac{2}{4-\sqrt{11}}$	e. $\frac{3}{2-\sqrt{6}}$	f. $\frac{7}{3+\sqrt{10}}$	g. $\frac{4}{4-\sqrt{3}}$	h. $\frac{2}{6+\sqrt{21}}$
i. $\frac{3}{1+\sqrt{9}}$	j. $\frac{4}{2-\sqrt{12}}$	k. $\frac{3}{6+\sqrt{6}}$	l. $\frac{15}{4-\sqrt{11}}$	m. $\frac{24}{5-\sqrt{7}}$	n. $\frac{39}{6+\sqrt{10}}$	o. $\frac{33}{5-\sqrt{3}}$	p. $\frac{64}{5-\sqrt{13}}$
q. $\frac{3}{5+\sqrt{25}}$	r. $\frac{16}{2-\sqrt{8}}$	s. $\frac{9}{15+\sqrt{9}}$	t. $\frac{14}{4-\sqrt{48}}$	u. $\frac{8}{6-\sqrt{90}}$	v. $\frac{10}{15+\sqrt{125}}$	w. $\frac{16}{9-\sqrt{45}}$	x. $\frac{14}{3-\sqrt{72}}$

Answers

7a. $\frac{2\sqrt{5}}{5}$ 7b. $\frac{4\sqrt{7}}{7}$ 7c. $\frac{7\sqrt{5}}{5}$ 7d. $\frac{9\sqrt{11}}{11}$ 7e. $\frac{11\sqrt{6}}{12}$ 7f. $\frac{7\sqrt{10}}{30}$ 7g. $\frac{4\sqrt{3}}{15}$ 7h. $\frac{6\sqrt{5}}{35}$ 7i. $\frac{\sqrt{6}}{3}$ 7j. $\frac{2\sqrt{3}}{3}$ 7k. $\frac{\sqrt{15}}{5}$ 7l. $\frac{3\sqrt{21}}{7}$ 7m. $\frac{\sqrt{6}}{4}$ 7n. $\frac{5\sqrt{6}}{4}$ 7o. $\frac{3\sqrt{3}}{5}$ 7p. $\frac{3\sqrt{14}}{35}$ 7q. $\frac{\sqrt{3}}{3}$ 7r. $\frac{2\sqrt{5}}{5}$ 7s. $\frac{3\sqrt{2}}{4}$ 7t. $\frac{9}{4}$ 7u. $\frac{3\sqrt{2}}{4}$ 7v. $\frac{7\sqrt{3}}{6}$ 7w. $\frac{3\sqrt{10}}{10}$ 7x. $\frac{1\sqrt{6}}{9}$ 8a. $-\frac{3(1-\sqrt{5})}{4}$ 8b. $-\frac{4(2+\sqrt{7})}{3}$ 8c. $3(2-\sqrt{3})$ 8d. $\frac{2(4+\sqrt{11})}{5}$ 8e. $-\frac{3(2+\sqrt{6})}{2}$ 8f. $-7(3-\sqrt{10})$ 8g. $\frac{4(4+\sqrt{3})}{13}$ 8h. $\frac{2(6-\sqrt{21})}{15}$ 8i. $\frac{3}{4}$ 8j. $-1-\sqrt{3}$ 8k. $\frac{6-\sqrt{6}}{10}$ 8l. $3(4+\sqrt{11})$ 8m. $\frac{4(5+\sqrt{7})}{3}$ 8n. $\frac{3(6-\sqrt{10})}{2}$ 8o. $\frac{3(5+\sqrt{3})}{2}$ 8p. $\frac{16(5+\sqrt{13})}{3}$ 8q. $\frac{3}{10}$ 8r. $-8(1+\sqrt{2})$ 8s. $\frac{1}{2}$ 8t. $-\frac{7(1+\sqrt{3})}{4}$ 8u. $-\frac{4(2+\sqrt{10})}{9}$ 8v. $\frac{3-\sqrt{5}}{2}$ 8w. $\frac{4(3+\sqrt{5})}{3}$ 8x. $-\frac{2(1+2\sqrt{2})}{3}$

Laws of exponents

We use exponents to denote repeated multiplication, e.g.,

$$a^3 = a \cdot a \cdot a.$$

It's easiest to remember the laws

$$(1)\qquad \begin{gathered} a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n \\ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad \frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}. \end{gathered}$$

by thinking of simple examples. For instance,

$$a^3 a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) = a^5,$$

and

$$(a^3)^2 = (a \cdot a \cdot a)^2 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) = a^6,$$

and

$$(ab)^3 = (ab)(ab)(ab) = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3 b^3,$$

are examples of the first three laws in (1).

The laws (1) are also true for exponents that are zero, negative or rational numbers after we define

$$(2)\qquad a^0 = 1 \qquad a^{-n} = \frac{1}{a^n} \qquad a^{m/n} = \sqrt[n]{(a^m)} = (\sqrt[n]{a})^m,$$

with some restrictions. For instance, we can't raise 0 to a negative exponent, since $\frac{1}{0}$ is undefined, and the last two equations are true only if the corresponding roots are defined. (Remember that we can't take an even root of a negative number.)

The most common mistake students make with the laws of exponents is to confuse $(ab)^n$ with $(a+b)^n$. In general, $(a+b)^n$ is **not** equal $a^n + b^n$.

Radicals

$\sqrt[n]{a}$ refers to a solution to the equation $x^n = a$, if one exists.

If n is odd, then every real number has exactly one nth root. For instance $\sqrt[3]{-8} = -2$, since -2 is the unique solution to the equation $x^3 = -8$.

If n is even, then negative numbers don't posses real nth roots, and every positive number has two real nth roots, one positive and one negative. In that case, the symbol $\sqrt[n]{a}$ stands for the **positive** nth root of a. The radical symbol $\sqrt{\ }$ with no index is always assumed to mean the square root $\sqrt[2]{\ }$.

Example 1:

Both 3 and -3 are solutions to $x^2 = 9$; both 3 and -3 are square roots of 9, but $\sqrt{9}$ stands for the *positive* square root, or 3.

end Example 1

Example 2:

The equation $x^2 = -9$ has no real solutions, so $\sqrt{-9}$ does not exist.

end Example 2

One of the reasons that we can think of $\sqrt[n]{a}$ as a power of a is the fact that

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

provided all three roots exist. That is,

$$(ab)^{1/n} = a^{1/n}b^{1/n},$$

as in (1). This is useful when we have to simplify expressions of the form $\sqrt[n]{a}$ where n is a positive integer and a is an integer. We look for the perfect nth powers that are factors of a, which we could always find by factoring a down to prime numbers.

Example 3: $\sqrt[3]{-48} = \sqrt[3]{-8\cdot 6} = \sqrt[3]{-8}\cdot\sqrt[3]{6} = -2\sqrt[3]{6}$.

end Example 3

Example 4: $\sqrt{72} = \sqrt{36\cdot 2} = \sqrt{36}\cdot\sqrt{2} = 6\sqrt{2}$.

If we hadn't noticed that 36 is the largest square factor of 72, we still could have simplified this radical completely by finding that the prime factorization of 72 is $2^3 3^2$. Then

$$\sqrt{72} = (3^2 2^3)^{1/2} = 3^{2/2}2^{3/2} = 3\cdot 2^{1+\frac{3}{2}} = 3\cdot 2^1\cdot 2^{1/2} = 6\sqrt{2}.$$

end Example 4

Example 5:
$\sqrt[4]{144} = \sqrt[4]{12^2} = \sqrt[4]{2^4 3^2} = 2\sqrt[4]{3^2}$. Since $\sqrt[4]{3^2} = 3^{2/4} = 3^{1/2}$, the answer can be simplified further to $2\sqrt[2]{3}$.

end Example 5

There's one important hitch in using the laws of exponents with fractional exponents. If $a \geq 0$, then both $(a^{1/n})^n$ and $(a^n)^{1/n}$ correctly simplify to a. However, if $a < 0$ and n is an even integer, $(a^{1/n})^n$ doesn't exist, and, because $\sqrt[n]{\ }$ refers to the *positive* nth root, $(a^n)^{1/n} = -a$, as in this example:

Example 6: $((-3)^2)^{1/2} = \sqrt{(-3)^2} = \sqrt{9} = 3$. *In particular, the result is not* -3.

end Example 6

The rule to remember in general is

$$\boxed{\text{If } n \text{ is an even integer, then } (a^n)^{1/n} = |a|.}$$

9. Simplify the radical or state that it does not exist.

a. $\sqrt{27}$ b. $-\sqrt{125}$ c. $\sqrt{108}$ d. $\sqrt{-800}$ e. $\sqrt{540}$ f. $\sqrt{32}$ g. $\sqrt{128}$ h. $\sqrt{405}$
i. $\sqrt{96}$ j. $\sqrt{2700}$ k. $\sqrt{80}$ l. $\sqrt{1125}$ m. $\sqrt{98}$ n. $\sqrt{363}$ o. $\sqrt{243}$ p. $\sqrt{9000}$

10. Simplify the radical or state that it does not exist.

a. $\sqrt[3]{135}$ b. $\sqrt[4]{176}$ c. $\sqrt[5]{-160}$ d. $\sqrt[2]{180}$ e. $\sqrt[6]{-64}$ f. $\sqrt[3]{-324}$ g. $\sqrt[4]{128}$
h. $\sqrt[5]{1458}$ i. $\sqrt[3]{20,000}$ j. $\sqrt[4]{405}$ k. $\sqrt[5]{-96}$ l. $\sqrt[2]{1125}$ m. $\sqrt[4]{-48}$ n. $\sqrt[3]{-378}$
o. $\sqrt{720}$ p. $\sqrt[6]{1280}$ q. $\sqrt[5]{128}$ r. $\sqrt[4]{-10,000}$ s. $\sqrt[3]{-80,000}$ t. $\sqrt[3]{320}$ u. $\sqrt[3]{500}$
v. $\sqrt[3]{-216}$ w. $\sqrt[5]{1024}$ x. $\sqrt{384}$ y. $\sqrt[3]{-54}$ z. $\sqrt[3]{648}$ 27. $\sqrt[6]{-128}$ 28. $\sqrt[3]{-128}$

11. Rewrite the expression in simplest radical form or state that it does not exist.

a. $9^{-3/2}$ b. $(-27)^{2/3}$ c. $(-27)^{5/3}$ d. $(27)^{-5/3}$ e. $(-27)^{-5/3}$ f. $(-81)^{5/2}$ g. $125^{-2/3}$

h. $16^{-3/4}$ i. $36^{3/2}$ j. $64^{-2/3}$ k. $81^{3/4}$ l. $125^{-4/3}$ m. $9^{5/2}$ n. $81^{-5/4}$

o. $32^{6/5}$ p. $8^{-2/3}$ q. $75^{1/2}$ r. $24^{2/3}$ s. $324^{3/4}$ t. $90^{-3/2}$ u. $(45)^{3/2}$

v. $(144)^{3/4}$ w. $108^{3/2}$ x. $108^{-2/3}$ y. $12^{3/2}$ z. $(-160)^{-3/5}$ 27. $(-96)^{3/4}$ 28. $32^{-3/4}$

29. $7^{5/4}$ 30. $250^{-3/2}$ 31. $(-500)^{2/3}$ 32. $600^{-2/3}$

12. Rewrite without parentheses. Express your answer without negative exponents, and again without fractions (other than exponents themselves). Assume all variables are positive.

a. $(x^3y^{-5})^2$ b. $(u^{-1/2}v^{1/3})^{-12}$ c. $y^{-5}(x^2y^2x^{-4})^3$

d. $(a^3bc^{-2}b^{-1}a^{-4})^{-1/2}$ e. $(uv^{-2}u^{-3}v^8)^{3/2}$ f. $(x(x^2y^{-2})^3yx^5)^{-1/3}$

g. $a^{-3}(ab^3)^{1/6}b^4$ h. $a^9b^{-2}c^4(4b)^{-2}(cb^2)^{-1}(2a^{-1}c^3)^3$ i. $2u^3(3v^2)^3wu^{10}3^{-1}$

j. $\sqrt{xy}(x^2y)^3(y^{-1}z)^4$ k. $(8x^2y)^{1/4}(2^{3/2}x^9y^4)^{1/6}$ l. $3^{-3}x^2y^3(9x^{1/3}y^{2/3})^2\sqrt[4]{xy}$

m. $(3(x+y))^2$ n. $2(u-v)^3(uv)^{-2}$ o. $ab^{-1}(1+ab)^2(1-ab)^2$

p. $\dfrac{(xy^2x^{-1}y)^{-2}}{(x^{-2}y^2)^2}$ q. $\dfrac{(x^3y^{-5})^2}{x^{-9}(y^3)^2}$ r. $\dfrac{(\frac{u}{v})(\frac{v^2}{u^3})}{\frac{3}{v}-\frac{7}{v}}$

s. $\dfrac{xy^2(x-y)^{-2}}{(x-y)^{-3}y^4}$ t. $\dfrac{v^2(\frac{uv^{-1}}{u^{-1}v})}{(\frac{u^2}{v^{1/2}})^{-1/3}}$ u. $\dfrac{a(b^3a^{-2})^2}{a^{-3}((b^{-4})^{-2/3})^{-1/2}}$

Answers

9a. $3\sqrt{3}$ 9b. $-5\sqrt{5}$ 9c. $6\sqrt{3}$ 9d. DNE 9e. $6\sqrt{15}$ 9f. $4\sqrt{2}$ 9g. $8\sqrt{2}$ 9h. $9\sqrt{5}$ 9i. $4\sqrt{6}$ 9j. $30\sqrt{3}$ 9k. $4\sqrt{5}$ 9l. $15\sqrt{5}$
9m. $7\sqrt{2}$ 9n. $11\sqrt{3}$ 9o. $9\sqrt{3}$ 9p. $30\sqrt{10}$ 10a. $3\sqrt[3]{5}$ 10b. $2\sqrt[4]{11}$ 10c. $-2\sqrt[5]{5}$ 10d. $6\sqrt[2]{5}$ 10e. DNE 10f. $-3\sqrt[3]{12}$ 10g. $2\sqrt[4]{8}$
10h. $3\sqrt[5]{6}$ 10i. $10\sqrt[3]{20}$ 10j. $3\sqrt[4]{5}$ 10k. $-2\sqrt[5]{3}$ 10l. $15\sqrt{5}$ 10m. DNE 10n. $-3\sqrt[3]{14}$ 10o. $12\sqrt{5}$ 10p. $2\sqrt[6]{20}$ 10q. $2\sqrt[5]{4}$
10r. DNE 10s. $-20\sqrt[3]{10}$ 10t. $4\sqrt[3]{5}$ 10u. $5\sqrt[3]{4}$ 10v. $-3\sqrt[3]{8}$ 10w. 4 10x. $8\sqrt{6}$ 10y. $-3\sqrt[3]{2}$ 10z. $6\sqrt[3]{3}$ 1027. DNE 1028. $-4\sqrt[3]{2}$
11a. $\frac{1}{27}$ 11b. 9 11c. -243 11d. $\frac{1}{243}$ 11e. $-\frac{1}{243}$ 11f. DNE 11g. $\frac{1}{25}$ 11h. $\frac{1}{8}$ 11i. 216 11j. $\frac{1}{16}$ 11k. 27 11l. $\frac{1}{625}$
11m. 243 11n. $\frac{1}{243}$ 11o. 64 11p. $\frac{1}{4}$ 11q. $5\sqrt{3}$ 11r. $4\sqrt[3]{9}$ 11s. $54\sqrt{2}$ 11t. $\frac{1}{270\sqrt{10}}$ 11u. $135\sqrt{5}$ 11v. $24\sqrt{3}$ 11w. $648\sqrt{3}$
11x. $\frac{1}{18\sqrt[3]{2}}$ 11y. $24\sqrt{3}$ 11z. $-\frac{1}{8\sqrt[5]{125}}$ 1127. DNE 1128. $\frac{1}{8\sqrt[4]{8}}$ 1129. $7\sqrt[4]{7}$ 1130. $\frac{1}{1250\sqrt{10}}$ 1131. $-50\sqrt[3]{2}$ 1132. $\frac{1}{20\sqrt[3]{45}}$
12a. $\frac{x^6}{y^{10}} = x^6y^{-10}$ 12b. $\frac{u^6}{v^4} = u^6v^{-4}$ 12c. $\frac{y}{x^6} = x^{-6}y$ 12d. $a^{1/2}c$ 12e. $\frac{v^9}{u^3} = u^{-3}v^9$ 12f. $\frac{y^{5/3}}{x^4} = x^{-4}y^{5/3}$ 12g. $\frac{b^{9/2}}{a^{-17/6}} =$ $a^{17/6}b^{9/2}$ 12h. $\frac{a^6c^{12}}{2b^6} = 2^{-1}a^6b^{-6}c^{12}$ 12i. $18u^{13}v^6w$ 12j. $\frac{x^{13/2}z^4}{y^{1/2}} = x^{13/2}y^{-1/2}z^4$ 12k. $2x^2y^{11/12}$ 12l. $3x^{35/12}y^{56/12}$
12m. $9x^2+18xy+9y^2$ 12n. $2uv^{-2}-6v^{-1}+6u^{-1}-2u^{-2}v = \frac{2u}{v^2}-\frac{6}{v}+\frac{6}{u}-\frac{2v}{u^2}$ 12o. $ab^{-1}-2a^3b+a^5b^3 = \frac{a}{b}-2a^3b+a^5b^3$
12p. $\frac{x^4}{y^{10}} = x^4y^{-10}$ 12q. $\frac{x^{15}}{y^{16}} = x^{15}y^{-16}$ 12r. $-\frac{v^2}{4u^2} = -4^{-1}u^{-2}v^2$ 12s. $\frac{x^2}{y^2}-\frac{x}{y} = x^2y^{-2}-xy^{-1}$ 12t. $u^{8/3}v^{1/6}$ 12u. $b^{22/3}$

Factoring polynomials

To factor an algebraic expression is to rewrite it in an equivalent form in which the last operation performed is multiplication. When factoring a polynomial,

1. Factor out any common factors.
2. Then try to factor the polynomial with one of these methods.
   a. If the polynomial has two terms, try using one of the **special products**.
   b. If the polynomial has three terms, try factoring it with **backwards-FOIL**. It also pays to be able to recognize the perfect square $a^2 + 2ab + b^2 = (a + b)^2$.
   c. If the polynomial has four terms, try factoring it by **grouping**.
3. Test whether you can factor the factors.

Example 7: $8x^3 - 12x^2 + 10x - 15$

Since the polynomial has four terms, we factor the first two terms and the last two and hope to find a common factor.

$$(8x^3 - 12x^2) + (10x - 15) = 4x^2(2x - 3) + 5(2x - 3) = (4x^2 + 5)(2x - 3).$$

end Example 7

Example 8: $5x^4y + 22x^3y + 8x^2y$

This polynomial has a common factor of x^2y:

$$5x^4y + 22x^3y + 8x^2y = x^2y(5x^2 + 22x + 8).$$

To factor the three-term polynomial, look for a factorization $(ax \pm b)(cx \pm d)$ that, when multiplied using FOIL, gives First $= 5x^2$, Last $= 8$, and Outside+Inside $= 22x$:

$$x^2y(5x^2 + 22x + 8) = x^2y(5x + 2)(x + 4).$$

end Example 8

The familiar difference of squares formula is just one of large number of **special products**:

$$\begin{aligned}
x^2 - y^2 &= (x - y)(x + y)\\
x^3 - y^3 &= (x - y)(x^2 + xy + y^2)\\
x^4 - y^4 &= (x - y)(x^3 + x^2y + xy^2 + y^3)\\
x^5 - y^5 &= (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)\\
x^6 - y^6 &= (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)\\
&\vdots
\end{aligned}$$

You can verify any of these yourself by carefully multiplying out the right side. When we replace y with $-y$ in the odd-lines, we also obtain

$$\begin{aligned}
x^3 + y^3 &= (x + y)(x^2 - xy + y^2)\\
x^5 + y^5 &= (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)\\
&\vdots
\end{aligned}$$

Example 9: $8u^3 + v^3 = (2u)^3 + v^3 = (2u + v)(4u^2 - 2uv + v^2)$

The three-term polynomial can't be factored further using integers.

end Example 9

Example 10: $x^4 - 16 = (x - 2)(x^3 + 2x^2 + 4x + 8)$

This polynomial hasn't been factored completely because the cubic polynomial factors further by grouping:

$$(x^3 + 2x^2) + (4x + 8) = x^2(x + 2) + 4(x + 2) = (x + 2)(x^2 + 4),$$

so

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4).$$

We also could have arrived at this factorization by using square-minus-square:

$$x^4 - 16 = (x^2)^2 - 4^2 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4).$$

end Example 10

The quadratic $x^2 + 4$ is said to be **irreducible**, meaning that it has no real zeros and, consequently cannot be factored further without complex numbers. The quadratic $4x^2 - 2x + 1$ is also irreducible, since $b^2 - 4ac = (-2)^2 - 4 \cdot 4 \cdot 1 = -12 < 0$, so $4u^2 - 2uv + v^2$ can't be factored without using complex numbers for coefficients. In fact, the quadratic factor that appears in the difference-of-cubes formula is *always* irreducible.

Every polynomial with real coefficients can be factored as a product of linear and irreducible quadratic factors with real coefficients (although there is no method that produces the exact factorization of *all* polynomials). If we allow ourselves to use complex coefficients, then every polynomials can be factored as a product of linear factors only. When we use only real coefficients in our factors, we're said to be factoring the polynomial **over the real numbers**. When we allow complex coefficients in our factors, we're said to be factoring the polynomial **over the complex numbers**.

13. Factor the polynomial over the real numbers.

a. $x^2 - 16$	b. $x^2 - 9$	c. $25 - x^2$	d. $8 - 2x^2$	e. $36 - u^2$
f. $4 - 9x^2$	g. $16x^2 - 9y^2$	h. $16x^3y - 36xy^3$	i. $4x^2 - 5$	j. $4x^2 + 5$
k. $8x^3 - y^3$	l. $x^5y^2 - 27x^2y^5$	m. $3u^4v + 81uv^4$	n. $40x^3 + 625y^3$	o. $x^6 + 125y^3$
p. $x^3y^3 - 8$	q. $x^4 - 16$	r. $x^4 - 81y^4$	s. $4x^4 - 16$	t. $128 - 18u^4$
u. $243u^4 - 375v^4$	v. $1 - 8x^3y^3$	w. $xy - 8x^3y^3$	x. $x^6 - y^6$	y. $x^6 + 64$

14. Factor the polynomial over the real numbers.

a. $2x^2 - x - 1$	b. $2x^2 - 11x + 12$	c. $2x^2 - 6x - 20$	d. $6x^2 - 13x + 5$
e. $4x^2 + 4x + 1$	f. $9x^2 - 6xy + y^2$	g. $8x^3 - 8x^2 + 2x$	h. $4x^2 + 12xy + 9y^2$
i. $15x^2 + 17x - 4$	j. $15x^2 - 11x - 4$	k. $-6x^2 + 8x + 8$	l. $x^2 + 3x - 28$
m. $2x^2 - 26x - 28$	n. $2x^2 - x - 28$	o. $3x^2 - 4xy - 15y^2$	p. $40x^3y + 40x^2y^2 + 10xy^3$
q. $5x^2 - 20x^3 + 20x^4$	r. $16x^4 + 24x^2 + 9$	s. $16x^4 - 8x^2 + 1$	t. $5x^2 + xy - 4y^2$
u. $3x^2 - 22xy + 24y^2$	v. $2x^2 - 23xy + 30y^2$	w. $3x^2 + 6xy - 24y^2$	x. $5x^2 - 30xy + 40y^2$
y. $6x^2 - 11xy^2 + 5y^4$	z. $15x^6 + 23x^3y^2 + 4y^4$	27. $3x^4 - 10x^2y + 8y^2$	28. $9x^4 + 38x^2y^2 + 8y^4$

15. Factor the polynomial over the real numbers.

a. $2x^3 - 3x^2 + 8x - 12$ b. $3x^3 + 6x^2 + 5x + 10$ c. $4x^5 - 2x^4 + 24x^3 - 12x^2$

d. $10x^4 - 22x^3 + 50x^2 - 110x$ e. $6x^3 - 8x^2y + 3xy^2 - 4y^3$ f. $20u^3 - 4u^2v + 5uv^2 - v^3$

g. $5x^3 - 3x^2 - 20x + 12$ h. $8x^3 + 4x^2 - 18x - 9$ i. $4x^3 - x^2 - 36x + 9$

j. $13u^3 - 12u^2v - 13uv^2 + 12v^3$ k. $2u^3 - 7u^2v - 32uv^2 + 112v^3$ l. $2x^3 - 3x^2 - 6x + 9$

m. $8x^3 + 4x^2 - 4x - 2$ n. $x^4 + 3x^3 - 8x - 24$ o. $27 - 81y + y^3 - 3y^4$

p. $x^5 - 9x^3 - 8x^2 + 72$ q. $4x^5 + x^3 + 108x^2 + 27$ r. $u^3v^2 - u - 3u^2v^2 + 3$

Answers

13a. $(x+4)(x-4)$ 13b. $(x-3)(x+3)$ 13c. $(5-x)(5+x)$ 13d. $2(2-x)(2+x)$ 13e. $(6-u)(6+u)$ 13f. $(2-3x)(2+3x))$ 13g. $(4x-3y)(4x+3y)$ 13h. $4xy(2x-3y)(2x+3y)$ 13i. $(2x-\sqrt{5})(2x+\sqrt{5})$ 13j. does not factor further 13k. $(2x-y)(4x^2+2xy+y^2)$ 13l. $x^2y^2(x-3y)(x^2+3xy+9y^2)$ 13m. $3vu(u+3v)(u^2-3uv+9v^2)$ 13n. $5(2x+5y)(4x^2-10xy+25y^2)$ 13o. $(x^2+5y)(x^4-5x^2y+25y^2)$ 13p. $(xy-2)(x^2y^2+2xy+4)$ 13q. $(x-2)(x+2)(x^2+4)$ 13r. $(x-3y)(x+3y)(x^2+9y^2)$ 13s. $4(x-\sqrt{2})(x+\sqrt{2})(x^2+2)$ 13t. $2(4-u\sqrt{3})(4+u\sqrt{3})(16+3u^2)$ 13u. $3(3u-5v)(3u+5v)(9u^2+25v^2)$ 13v. $(1-2xy)(1+2xy+4x^2y^2)$ 13w. $xy(1-xy2\sqrt{2})(1+xy2\sqrt{2})$ 13x. $(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$ 13y. Has no real zeros; hence, does not factor without using complex coefficients. 14a. $(2x+1)(x-1)$ 14b. $(2x-3)(x-4)$ 14c. $2(2x+2)(x-5)$ 14d. $(3x-5)(2x-1)$ 14e. $(2x+1)^2$ 14f. $(3x-y)^2$ 14g. $2x(2x-1)^2$ 14h. $(2x+3y)^2$ 14i. $(3x+4)(5x-1)$ 14j. $(15x+4)(x-1)$ 14k. $-2(3x+2)(x-2)$ 14l. $(x-4)(x+7)$ 14m. $2(x+1)(x-14)$ 14n. $(2x+7)(x-4)$ 14o. $(3x+5y)(x-3y)$ 14p. $10xy(2x+y)^2$ 14q. $5x^2(1-2x)^2$ 14r. $(4x^2+3)^2$ 14s. $(2x-1)^2(2x+1)^2$ 14t. $(5x-4y)(x+y)$ 14u. $(x-6y)(3x-4y)$ 14v. $(2x-3y)(x-10y)$ 14w. $3(x+4y)(x-2y)$ 14x. $5(x-4y)(x-2y)$ 14y. $(6x-5y^2)(x-y^2)$ 14z. $(3x^3+4y^2)(5x^3+y^2)$ 1427. $(3x^2-4y)(x^2-2y)$ 1428. $(x^2+4y^2)(9x^2+2y^2)$ 15a. $(x^2+4)(2x-3)$ 15b. $(3x^2+5)(x+2)$ 15c. $x^2(x^2+6)(4x-2)$, or, $2x^2(x^2+6)(2x-1)$ 15d. $2x(x^2+5)(5x-11)$ 15e. $(2x^2+y^2)(3x-4y)$ 15f. $(4u^2+v^2)(5u-v)$ 15g. $(x-2)(x+2)(5x-3)$ 15h. $(2x+3)(2x-3)(2x+1)$ 15i. $(x-3)(x+3)(4x-1)$ 15j. $(u-v)(u+v)(13u-12v)$ 15k. $(u-4v)(u+4v)(2u-7v)$ 15l. $(x+\sqrt{3})(x-\sqrt{3})(2x-3)$ 15m. $(2x-\sqrt{2})(2x+\sqrt{2})(2x+1)$, or $2(x\sqrt{2}-1)(x\sqrt{2}+1)(2x+1)$ 15n. $(x+3)(x-2)(x^2+2x+4)$ 15o. $(1-3y)(3+y)(9-9y+y^2)$ 15p. $(x-2)(x-3)(x+3)(x^2+2x+4)$ 15q. $(x+3)(x^2-3x+9)(4x^2+1)$ 15r. $(uv-1)(uv+1)(u-3)$

Solving polynomial equations

One reason we factor polynomials is to find their **zeros**, i.e., the x-values that cause the polynomial to equal zero. We rely heavily on a special property of the number zero. For the product of two numbers to be zero, it is both necessary and sufficient that one of the two must be zero. That is,

$$\boxed{ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0.}$$

It is important to remember that no number other than zero has this property. For instance, $ab = 3$ does *not* mean that a or b has to equal 3.

To solve a polynomial equation, try to

$$\boxed{\text{Get zero on one side and factor the other.}}$$

Example 11: Solve for x in the equation $0 = 6x^2 + x - 1$

Factor the right side. $6x^2 + x - 1 = (2x + 1)(3x - 1)$. So, for this product to equal zero, one factor or the other must be zero:

$$\begin{aligned} 2x + 1 &= 0 & 3x - 1 &= 0 \\ 2x &= -1 & 3x &= 1 \\ x &= -1/2 & x &= 1/3 \end{aligned}$$

That is, $0 = 6x^2 + x - 1$ has two solutions: $x = -1/2$ and $x = 1/3$.

end Example 11

Example 12: Solve for x in the equation $4x^3 + 14x^2 = 8x$.

Get a zero on one side by subtracting $8x$ to both sides:

$$4x^3 + 14x^2 - 8x = 0.$$

Now factor the left side. First factor out the common factor of x, and then factor the three term polynomial using backwards FOIL:

$$2x(2x^2 + 7x - 4) = 2x(2x - 1)(x + 4) = 0.$$

For the product of $2x$, $(2x - 1)$, and $(x + 4)$ to equal zero, one of these factors has to be zero:

$$\begin{aligned} 2x &= 0 & 2x - 1 &= 0 & x + 4 &= 0 \\ x &= 0 & x &= 1/2 & x &= -4 \end{aligned}$$

That is, the solutions to $4x^3 + 14x^2 - 8x = 0.$ are $x = 0$, $1/2$, and -4.

end Example 12

The problem of find y-intercepts usually boils down to solving an equation, as the next example shows.

Example 13: Find the x- and y-intercepts along the curve $y = 6x^2 - x - 2$.

To find the y-intercepts, we set $x = 0$ and calculate $y = 6 \cdot 0^2 - 0 - 2 = -2$. That it, $(0, -2)$ is the y-intercept.

To find the x-intercepts, set $y = 0$ and solve for x by factoring:

$$0 = 6x^2 - x - 2 = (3x - 2)(2x + 1).$$

The solutions are

$$\begin{aligned} 3x - 2 &= 0 & 2x + 1 &= 0 \\ 3x &= 2 & 2x &= -1 \\ x &= 2/3 & x &= -1/2 \end{aligned}$$

so this curve has two x-intercepts: $(2/3, 0)$ and $(-1/2, 0)$

end Example 13

16. Find the zeros of the polynomial.

a. $x^2 - 16$ b. $x^2 - 9$ c. $25 - x^2$ d. $8 - 2x^2$ e. $36 - u^2$

f. $4 - 9x^2$ g. $16x^2 - 9$ h. $16x^3 - 36x$ i. $4x^2 - 5$

17. Solve for x.

a. $2x^2 - x - 1 = 0$ b. $2x^2 - 11x + 12 = 0$ c. $2x^2 - 6x - 20 = 0$ d. $6x^2 - 13x + 5 = 0$

e. $4x^2 + 4x = -1$ f. $9x^2 - 24x = -16$ g. $8x^3 - 8x^2 = -2x$ h. $4x^2 + 12x = -9$

i. $15x^2 + 17x - 4 = 0$ j. $15x^2 - 11x = 4$ k. $-6x^2 + 8x = -8$ l. $x^2 + 3x - 28 = 0$

Answers

16a. $x = \pm 4$ 16b. $x = \pm 3$ 16c. $x = \pm 5$ 16d. $x = \pm 2$ 16e. $u = \pm 3$ 16f. $x = \pm 2/3$ 16g. $x = \pm 3/4$ 16h. $x = 0$ or $\pm\, 3/2$ 16i. $x = \pm\sqrt{5}/2$ 17a. $x = 1, -1/2$ 17b. $x = 3/2, 4$ 17c. $x = 5, -1$ 17d. $x = 5/3, 1/2$ 17e. $x = -1/2$ 17f. $x = 4/3$ 17g. $x = 0, 1/2$ 17h. $x = -3/2$ 17i. $x = 1/5, -4/3$ 17j. $x = 1, -4/15$ 17k. $x = 2, -2/3$ 17l. $x = 4, -7$

Rational functions

A rational function is a fraction whose numerator and denominator are both polynomials. We perform arithmetic on rational functions just as with any fractions.

Simplification isn't something you do in your last step to satisfy your math professor; it's something you do at every step to make your work easier. Be on the lookout for common **factors** of the numerator and denominator and cancel these.

Example 14: $\frac{x+2}{x^2-4} = \frac{x+2}{(x-2)(x+2)} = \frac{1}{x-2}$

Notice that $\frac{1}{x-2}$ is considerably simpler than $\frac{x+2}{x^2-4}$. Canceling the $x+2$ was a good move.

end Example 14

Example 15: *Can I cancel the x in $\frac{x}{x+2}$?*

$\frac{x}{x+2}$ has no common factors, at least as it is written at present. You *can* cancel the x from top and bottom if you factor it out first, but the result might not be very useful:

$$\frac{x}{x+2} = \frac{x}{x\left(1+\frac{2}{x}\right)} = \frac{1}{1+\frac{2}{x}}.$$

end Example 15

18. Simplify the rational function.

a. $\dfrac{5x^2+22x+8}{x^2+6x+8}$ b. $\dfrac{8x^2+2x-1}{2x^2+3x+1}$ c. $\dfrac{2x^2-14x+20}{6x^2-8x-8}$

d. $\dfrac{12x^2+11x-15}{3x^2-4x-15}$ e. $\dfrac{2x^2+14x+20}{2x^2+6x-20}$ f. $\dfrac{2x^2+7x+3}{4x^2+4x+1}$

g. $\dfrac{9x^2-12x+4}{3x^2-20x+12}$ h. $\dfrac{49x^2+28x+4}{21x^2-36x-12}$ i. $\dfrac{x^3+2x^2-9x-18}{x^3-3x^2+4x-12}$

j. $\dfrac{2x^3+4x^2+x+2}{x^3+2x^2-4x-8}$ k. $\dfrac{x^3-8}{x^2-4}$ l. $\dfrac{4x^2-1}{8x^3+1}$

m. $\dfrac{4x^2-4x+1}{4x^2-1}$ n. $\dfrac{9x^2+1}{9x^2+6x+1}$ o. $\dfrac{x^4-16}{2x^3+5x^2+8x+20}$

p. $\dfrac{6x^3+9x^2+2x+3}{15x^3-27x^2+5x-9}$ q. $\dfrac{6x^3-2x^2+3x-1}{6x^3+2x^2+3x+1}$ r. $\dfrac{3x^3+3x^2+4x+4}{3x^3+3x^2-4x-4}$

s. $\dfrac{x^3+x^2-x-1}{x^2+2x+1}$ t. $\dfrac{27x^3-9x^2-3x+1}{9x^2-6x+1}$ u. $\dfrac{x^2-5x+6}{x^4-13x^2+36}$

19. Perform the operation indicated and simplify the resulting rational function.

a. $\dfrac{2}{x-3} - \dfrac{1}{2x+1}$ b. $\dfrac{1}{x-5} + \dfrac{2}{5x-3}$ c. $\dfrac{3}{x-3} - \dfrac{4}{x-1}$

d. $\dfrac{2}{3x+1} + \dfrac{2}{3x-1}$ e. $\dfrac{1}{x+4} - \dfrac{1}{x-4}$ f. $\dfrac{2}{x^2+3} - \dfrac{1}{3x+1}$

g. $\dfrac{2x}{3x^2-23x-8} - \dfrac{1}{3x^2+13x+4}$ h. $\dfrac{2x+1}{2x-1} - \dfrac{3x-1}{3x+1}$ i. $\dfrac{2}{x+3} - \dfrac{1}{3x^2+8x-3}$

j. $\dfrac{1}{2x^2-11x+15} + \dfrac{1}{2x^2-9x+10}$ k. $\dfrac{3}{x^2+4} - \dfrac{4}{x+2}$ l. $\dfrac{x+3}{2x^2+3x-2} + \dfrac{1}{x^2-x-6}$

m. $\dfrac{1}{2x^3-3x^2+10x-15} - \dfrac{x}{8x^3-12x^2}$ n. $\dfrac{1}{x^2-4} - \dfrac{1}{x^3-8}$ o. $\dfrac{1}{(x^2-4)(x+1)} + \dfrac{1}{x^2-4}$

p. $\dfrac{3x}{6x^3-3x^2+2x-1} + \dfrac{4x}{2x^3-x^2+16x-8}$ q. $\dfrac{x^3}{x+1} - \dfrac{1}{x^3+1}$ r. $\dfrac{2x}{x^2+9} + \dfrac{1}{x^4+6x^2-27}$

s. $\dfrac{x-1}{x^2+x-2} - \dfrac{4-x}{x^3+2x^2-4x-8}$ t. $\dfrac{1}{x^2-2x} - \dfrac{1}{2x-4}$ u. $\dfrac{2x+1}{10x^2-7x-6} + \dfrac{x+2}{5x^2+4x-12}$

v. $\dfrac{2x-1}{10x^2-17x+6} + \dfrac{x+2}{5x^2+16x+12}$ w. $\dfrac{2x}{x^2+4} - \dfrac{1}{x+2}$ x. $\dfrac{1}{x^3-x^2-9x+9} - \dfrac{1}{2x^2-18}$

y. $\dfrac{1}{2x^3+5x^2-2x-10} - \dfrac{1}{3x^2-3}$ z. $\dfrac{3}{x+4} - \dfrac{4}{x^2-16}$ 27. $\dfrac{1}{x^2-3x-4} - \dfrac{1}{2x^2-11x+12}$

20. Perform the operation indicated and simplify the resulting rational function.

a. $\dfrac{x+1}{x-3} \cdot \dfrac{x+2}{x-4}$ b. $\dfrac{2}{x+2} \cdot \dfrac{6}{x+2}$ c. $\dfrac{x+3}{2x+1} \cdot \dfrac{x-3}{x^2+2x-3}$

d. $\dfrac{x^2-4}{2x^2-5x+3} \cdot \dfrac{x^2-3x+2}{x^2+11x+18}$ e. $\dfrac{x+3}{2x^2-3x-20} \cdot \dfrac{2x+5}{x^2-7x+12}$ f. $\dfrac{x^2-4}{x^2-9} \div \dfrac{x-2}{x+3}$

g. $\dfrac{2x^2-x-1}{x^2+4x-5} \cdot \dfrac{x^2-25}{8x^2-2x-3}$ h. $\dfrac{2x^2-5x+3}{15x^3+5x^2} \div \dfrac{8x^2-12x}{3x^2+7x+2}$ i. $\dfrac{\frac{x}{x+3} - \frac{2}{5}}{\frac{x-2}{x+3}}$

j. $\dfrac{x^3+3x^2-4x-12}{x^3+3x^2+4x+12} \div \dfrac{x^4+9x^2+20}{3x-6}$ k. $\dfrac{\frac{x}{x-4} + \frac{1}{3}}{x^2-1}$ l. $\dfrac{\frac{2x+1}{x} - \frac{7}{3}}{\frac{x-3}{x^2}}$

m. $\dfrac{\frac{2x+3}{3x-4} + 5}{x-1}$ n. $\dfrac{\frac{x+1}{x-2} - \frac{x-1}{x+2}}{\frac{x+2}{x+3} - \frac{x-2}{x-3}}$ o. $\dfrac{\frac{2x-1}{x-1} - \frac{x+4}{x}}{x-2}$

Answers

18a. $\frac{5x+2}{x+2}$ 18b. $\frac{4x-1}{x+1}$ 18c. $\frac{x-5}{3x+2}$ 18d. $\frac{4x-3}{x-3}$ 18e. $\frac{x+2}{x-2}$ 18f. $\frac{x+3}{2x+1}$ 18g. $\frac{3x-2}{x-6}$ 18h. $\frac{7x+2}{3x-6}$ 18i. $\frac{(x+2)(x+3)}{x^2+4}$ 18j. $\frac{2x^2+1}{x^2-4}$
18k. $\frac{x^2+2x+4}{x+2}$ 18l. $\frac{2x-1}{4x^2-2x+1}$ 18m. $\frac{2x-1}{2x+1}$ 18n. does not simplify further 18o. $\frac{x^2-4}{2x+5}$ 18p. $\frac{2x+3}{5x-9}$ 18q. $\frac{3x-1}{3x+1}$ 18r. $\frac{3x^2+4}{3x^2-4}$
18s. $x-1$ 18t. $3x+1$ 18u. $\frac{1}{x^2+5x+6}$ 19a. $\frac{3x+5}{(x-3)(2x+1)}$ 19b. $\frac{7x-13}{(x-5)(5x-3)}$ 19c. $\frac{9-x}{(x-3)(x-1)}$ 19d. $\frac{12x}{9x^2-1}$ 19e. $-\frac{8}{x^2-16}$
19f. $\frac{-x^2+6x-1}{(x^2+3)(3x+1)}$ 19g. $\frac{2x^2+7x+8}{(3x+1)(x+4)(x-8)}$ 19h. $\frac{10x}{(2x-1)(3x+1)}$ 19i. $\frac{6x-3}{(3x-1)(x+3)}$ 19j. $\frac{1}{(x-2)(x-3)}$ 19k. $\frac{-4x^2+3x-10}{(x^2+4)(x+2)}$
19l. $\frac{x^2+x-7}{(x-3)(x+2)(2x-1)}$ 19m. $\frac{-x^2+4x-5}{4x(2x-3)(x^2+5)}$ 19n. $\frac{x^2+x+2}{(x-2)(x+2)(x^2+2x+4)}$ 19o. $\frac{1}{(x-2)(x+1)}$ 19p. $\frac{15x^3+28x}{(2x-1)(x^2+8)(3x^2+1)}$ 19q. $\frac{x^5-x^4+x^3-1}{x^3+1}$
19r. $\frac{2x^3-6x+1}{(x^2+9)(x^2-3)}$ 19s. $\frac{x^2+x-8}{(x-2)^2(x+2)}$ 19t. $-\frac{1}{2x}$ 19u. $\frac{2}{5x-6}$ 19v. $\frac{10x}{25x^2-36}$ 19w. $\frac{x^2+4x-4}{(x^2+4)(x+2)}$ 19x. $-\frac{1}{2(x+3)(x-1)}$
19y. $\frac{-2}{3(2x+5)(x-1)}$ 19z. $\frac{3x-16}{x^2-16}$ 1927. $\frac{1}{(x+1)(2x-3)}$ 20a. $\frac{(x+1)(x-3)}{(x+2)(x-4)}$ 20b. $\frac{12}{x^2+4x+4}$ 20c. $\frac{x-3}{(2x+1)(x-1)}$ 20d. $\frac{(x-2)^2}{(2x-3)(x+9)}$
20e. $\frac{x+3}{(x-4)^2(x-3)}$ 20f. $\frac{x+2}{x-3}$ 20g. $\frac{x-5}{4x-3}$ 20h. $\frac{(x-1)(x+2)}{20x^3}$ 20i. $\frac{3}{5}$ 20j. $\frac{1}{3}(x+2)(x^2+5)$ 20k. $\frac{4}{x+1}$ 20l. $-\frac{x}{3}$ 20m. $\frac{17}{3x-4}$
20n. $\frac{3(x^2-9)}{x^2-4}$ 20o. $\frac{x-2}{x^2-x}$