Interest

Math 103.001, Maymester 2006

Present Value and Future Value

Example 1 . . . .
If you deposited \$300 today is an account that pays interest at an **nominal annual rate (NAR)** *of 12% compounded quarterly, how much would be in your account in 5 years?*

Compounding quarterly means that the bank pays you interest not just once a year but once every quarter, or four times a year. Since the stated NAR is 12%, the quarterly rate is one fourth of this, or $\frac{1}{4} * 12\% = 3\%$.

Let's write a balance sheet for your account. Your initial balance is \$300, so the sheet starts off like this:

Time	Balance
now	\$300.00

After your \$300.00 has been in the account for one quarter, the bank pays you interest in the amount of 3% of \$300.00, or $.03 * \$300.00 = \$9.00$. So after one quarter, the balance sheet looks like this:

Time	Balance
now	\$300.00
after 1 qr.	\$309.00

At the end of the second quarter, the bank again adds interest to your account, but this time, it's 3% of the \$309.00 that's been in your account for the past quarter. That is, you earn $.03 * \$309.00 = \$9.27$ in interest. Adding this to your \$309.00 results in the following balance sheet:

Time	Balance
now	\$300.00
after 1 qr.	\$309.00
after 2 qr.	\$318.27

To solve this problem, we could continue this way for 18 more quarters,

Time	Balance
now	\$300.00
after 1 qr.	\$309.00
after 2 qr.	\$318.27
after 3 qr.	\$327.82
after 4 qr.	\$337.65
after 5 qr.	\$347.78

etc. (always rounding to the nearest penny), but there's an easier way.

There's actually a pattern in the right hand column of the balance sheet, although it's a little hard to recognize as the numbers are currently written. Let's rewrite it as follows.

Remember how we got the \$309.00? That was \$300.00 plus \$9.00 interest, and the \$9.00 came by taking 3% of \$300.00. In symbols,

$$309 = 300 + .03 * 300.$$

Factor out the 300 on the right and you get

$$309 = 300(1 + .03)$$

or

$$309 = 300(1.03)$$

Similarly, the \$318.27 came from increasing \$309.00 by 3%:

$$\begin{aligned} 318.27 &= 309.00 + .03 * 309.00 \\ &= 309.00(1.03) \end{aligned}$$

This points to an important fact. **To increase a number by 3% is the same as multiplying the number by 1.03.** Therefore each dollar amount in the balance sheet is 1.03 times the previous amount.

Time	Balance
now	\$300.00
after 1 qr.	$\$309.00 = 300.00(1.03)$
after 2 qr.	$\$318.27 = 309.00(1.03)$
after 3 qr.	$\$327.82 = 318.27(1.03)$
after 4 qr.	$\$337.65 = 327.82(1.03)$
after 5 qr.	$\$347.78 = 337.65(1.03)$
	etc.,

What's more, since $309 = 300(1.03)$, that means

$$\begin{aligned} \$318.27 &= \$309.00(1.03) \\ &= (\$300.00(1.03))(1.03) \\ &= \$300.00(1.03)^2 \end{aligned}$$

and

$$\begin{aligned} \$327.82 &= \$318.27(1.03) \\ &= (300(1.03)^2)(1.03) \\ &= 300(1.03)^3 \end{aligned}$$

so the balance sheet could be written

Time	Balance
now	\$300.00
after 1 qr.	$300.00(1.03)$
after 2 qr.	$300.00(1.03)^2$
after 3 qr.	$300.00(1.03)^3$
after 4 qr.	$300.00(1.03)^4$
after 5 qr.	$300.00(1.03)^5$

The pattern is that, after any number of quarters, the balance is \$300.00 times 1.03 raised to the number of quarters. More compactly, the balance after n quarters will be $\$300.00(1.03)^n$. Therefore the balance after 20 quarters (= 5 years) will be

$$\$300.00(1.03)^{20} = \$541.83.$$

. . . . *End of Example 1*

In these circumstances, we say that \$541.83 is the **future value** of \$300, and that the **present value** of \$541.83 is \$300. Whenever we use the words present or future value, it must be in reference to some particular interest rate, compounding scheme, and length of investment.

The length of time between interest payments is called the compounding **period**, and rate at which the bank pays interest in one period is called the **periodic interest rate**. In Example 1, the period was a quarter year (three months) and periodic interest rate was 3%. The periodic interest rate is always the NAR divided by the number of interest periods in one year. In general, the relationship between future and present values is

$$\text{(future value)} = \text{(present value)}(1 + \text{periodic interest rate})^{\text{number of periods}}$$

or, more simply,

$$\boxed{F = P(1+i)^n}$$

The variables in these (and all future) equations are:

$$\begin{aligned}
P &= \text{Present value} \\
F &= \text{Future value} \\
R &= \text{Rent (to be discussed later)} \\
i &= \text{periodic interest rate} \\
n &= \text{length of investment time, measured in periods}
\end{aligned}$$

Example 2 . . . .
How much money should you deposit today in an account paying NAR 3% compounded monthly so that, after five years, your balance will be \$500.00?

In other words, what is the present value of \$500.00 in five years under this interest scheme? Since \$500 is the balance in the future, $F = \$500$. The number of interest periods, in this case months, in five years is $n = 5 * 12 = 60$, and the interest rate per period is $.03 \div 12 = .0025$. To find P, we solve the equation

$$500 = P(1.0025)^{60}$$

so that

$$P = \frac{500}{(1.0025)^{60}} = 430.43.$$

. . . . *End of Example 2*

Solving for P in the F vs. P formula above is equivalent to using the formula

$$\boxed{P = \frac{F}{(1+i)^n}}$$

Effective Annual Yield

Effective annual yield is a concept used to compare bank accounts with different interest rates and compounding schemes.

Example 3 . . . .
Bank A offers its depositors an NAR of 3% compounded semiannually, while Bank B pays NAR 3.25% compounded quarterly. Which is the better deal for depositors?

To answer that question, imagine depositing one dollar in each account. The balance over your original dollar is the **effective annual yield** for that account.

One dollar in Bank A, left alone for one year, will turn into

$$1\left(1+\frac{.03}{2}\right)^2 = 1.030225$$

while a dollar in Bank B will grow to

$$1\left(1+\frac{.0325}{4}\right)^4 = 1.032898243\ldots$$

The effective annual yield at Bank A is 3.0225%, while at Bank B the effective annual yield is 3.2898243%. Bank B is a better deal for depositors.
. . . . *End of Example 3*

Important note: Don't round these both to 3%! There is a difference between these two annual rates that will show up when we invest a larger amount for a longer time. Note also that we don't need a special formula to calculate effective annual yield; just calculate the future value of one dollar deposited in each bank for one year, and the effective annual yield will be the amount added to your original dollar.

We say that the **nominal** interest rate at Bank A is 3%, to distinguish it from the effective annual yield. The reason for calling 3.0225% the *effective* rate of interest is that, provided you keep your money in the bank for a whole number of years, an account that pays NAR 3% compounded semiannually will pay exactly the same as an account that pays NAR 3.0225% compounded annually.

For instance, if I deposit \$1000 in Bank A for 7 years, I'll end up with

$$\$1000\left(1+\frac{.03}{2}\right)^{14} = \$1231.76,$$

and if I deposit \$1000 in a bank paying NAR 3.0225% compounded annually for 7 years, I'll end up with exactly the same amount,

$$\$1000(1.030225)^7 = \$1231.76.$$

Typically, when a bank is trying to attract depositors, it advertises its effective annual yield, and when it's trying to attract borrowers, it advertises its NAR.

Average Rate of Growth

When something we purchase as an investment, such as stock or property, grows (or declines) in value at a variable rate, we can measure its performance by calculating its **average rate of growth**.

Example 4 . . . .
Suppose a share of stock was worth \$100 on Jan. 1, 1995, and \$175 on Jan. 1, 2000. Find its annual average rate of growth over this time period.

You should think of the average annual rate of growth as the NAR i a bank would have to pay, compounding once a year, in order for \$100 to grow to \$175 over the five years in question. Set $P = \$100$, $F = \$175$, $n = 5$, and solve for i in the equation

$$\$175 = \$100(1 + i)^5.$$

Divide by \$100:

$$1.75 = (1 + i)^5.$$

To get at i, we have to get rid of the exponent. Raise both sides to the power $\frac{1}{5}$ and use the rule that $(x^a)^b = x^{ab}$

$$1.75^{\frac{1}{5}} = \left((1 + i)^5\right)^{\frac{1}{5}} = 1 + i.$$

Therefore

$$i = 1.75^{\frac{1}{5}} - 1 = .11842691472\ldots$$

That is, the stock grew at the average rate of approximately 11.8% per year. A investment in this stock between Jan 1, 1995 and Jan 1, 2000 would have resulted in the same outcome as a deposit in a bank paying 11.842. . . % effective annual yield.
. . . . *End of Example 4*

In, general if we solve for i in the P-to-F formula, the result is

$$\boxed{i = \left(\frac{F}{P}\right)^{\frac{1}{n}} - 1}$$

If the investment falls in value, then its average annual rate of growth is negative, and could be called its rate of decline.

Example 5 . . . .
A share of stock, once worth \$30, is worth only \$21 three years later. Find its avarage rate of decline.

Using $F = 21$ and $P = 30$ and $n = 3$, we find

$$i = \left(\frac{21}{30}\right)^{\frac{1}{3}} - 1 = -.112095998\ldots$$

and therefore the stock fell in value an average of approximately 11.2% per year.
. . . . *End of Example 5*

Rent

We'll use the word **Rent** to describe any amount that is paid (to a creditor) or invested (in a bank account) each interest period. Rent will be denoted by the letter R.

Example 6 . . . .
Invest \$500 on the first of each month for 24 months, and what will you have in your account after making your last deposit? Suppose your account pays NAR 8.04%, compounded monthly.

To find the answer, figure out how much each \$500 deposit will grow by the time the two years are over. The periodic interest rate is $.0804 \div 12 = .0067$.

- The first \$500 deposited will be in your account for 23 months by the time we check the balance. It therefore contributes $500(1.0067)^{23}$ to your final balance.
- The second \$500 will be in your account for 22 months, contributing $500(1.0067)^{22}$ to the final balance.

Continuing along these lines,

- the second-to-last deposit contributes $500(1.0067)$ to your final balance, and
- the last deposit contributes 500.

Therefore, the total in your account after making the last deposit will be

$$\begin{aligned} &500(1.0067)^{23} + 500(1.0067)^{22} + \cdots + 500(1.0067)^{2} + 500(1.0067) + 500 \\ &= 500(1.0067^{23} + 1.0067^{22} + \cdots + 1.0067^{2} + 1.0067 + 1) \end{aligned}$$

Using a factoring formula from algebra, we can calculate the sum inside the parentheses with calculating its 24 terms.

As you will recall,

$$(x-1)(x+1) = x^2 - 1$$

and

$$(x-1)(x^2+x+1) = x^3 - 1.$$

In case you'd forgotten the second of these formulas, you can verify it by multiplying out the left side and obtaining the right:

$$\begin{aligned} (x-1)(x^2+x+1) &= x^3 + x^2 + x \\ &\quad - x^2 - x - 1 \\ &= x^3 - 1. \end{aligned}$$

In the same way, one could multiply out

$$\begin{aligned}(x-1)(x^3+x^2+x+1) &= x^4+x^3+x^2+x \\ &\qquad -x^3-x^2-x-1 \\ &= x^4-1\end{aligned}$$

to obtain the rule

$$(x-1)(x^3+x^2+x+1) = x^4-1.$$

Similarly,

$$(x-1)(x^4+x^3+x^2+x+1) = x^5-1$$

and

$$(x-1)(x^5+x^4+x^3+x^2+x+1) = x^6-1.$$

As you can probably guess, the rule is

$$(x-1)(x^{n-1}+x^{n-2}+\cdots+x^2+x+1) = x^n-1.$$

If we divide by $x-1$, the result is

$$x^{n-1}+x^{n-2}+\cdots+x^2+x+1 = \frac{x^n-1}{x-1}.$$

Getting back to Example 6, if $x = 1.0067$ and $n = 24$, this says

$$1.0067^{23}+1.0067^{22}+\cdots+1.0067^2+1.0067+1 = \frac{1.0067^{24}-1}{1.0067-1}$$

so the total amount in you account after making the last deposit in this example is

$$\$500\left(\frac{1.0067^{24}-1}{1.0067-1}\right) = \$12,971.67.$$

....*End of Example 6*

Under these circumstances, we say that the future value of these 24 payments is \$12,971.67.

In general, the future value F of n rent payments of size R is the amount that could be gotten by depositing those payments into a bank account and letting the interest accumulate:

$$\boxed{F = \frac{R((1+i)^n-1)}{i}}$$

Example 7 . . . .
I want to save up \$25,000 in 20 years by making monthly deposits into an account that pays NAR 9.96% compounded monthly. How much must I invest each month?

In other words, what must R be for the future value of these deposits to be \$25,000? Since $i = .0996 \div 12 = .0083$, the formula above implies that

$$25000 = \frac{R(1.0083^{240} - 1)}{.0083}$$

so, solving for R,

$$R = \frac{25000(.0083)}{1.0083^{240} - 1} = 33.09$$

. . . . *End of Example 7*

Solving for R in the F vs. R formula is the same as using this formula:

$$\boxed{R = \frac{Fi}{((1+i)^n - 1)}}$$

It is remarkable but correct that we need to invest so very little regularly to acquire such a large sum: our total deposits are only $240 * \$33.09 = \$7941.60$, so the total interest we earned over the twenty years must have been

$$\$25,000 - \$7941.60 = \$17,058.40.$$

We would have had to invest even less provided we were able to put it all up front at the beginning of the twenty years. To see exactly how much, calculate the present value of \$25,000:

$$P = \frac{25000}{1.0083^{240}} = 3438.71.$$

That is, \$3438.71 invested today at 9.96% compounded monthly will turn into \$25,000 in twenty years.

Amortized Loans

To amortize a loan means to figure out how the loan can be paid back exactly, with interest, in equal installments.

Example 8 . . . .
If I borrow \$8000.00 today to be repaid in 60 equal monthly payments, what should be the size of my monthly payments? Assume that my bank charges interest me at NAR 18% compounded monthly, and that my first payment if due one month from today.

First of all, the answer is *not* $\$8000 \div 60$, since my debt grows with interest, specifically at the monthly rate of $.18 \div 12 = .015$. Even though, in this problem, I pay interest to the bank and not vice-versa, the rules of calculating interest are the same. For instance, if I didn't make *any* payments to the bank, then after 5 years (60 months) my original debt of $P = \$8000$ would grow to

$$F = \$8000(1.015)^{60} = \$19,545.76.$$

One way to discover the size of my monthly payment is to rephrase the question as follows. What amount, deposited monthly for 60 months, would grow to $F = \$19,545.76$ on the day of my last deposit (which I could then use to eliminate my debt to the bank)? This is the same sort of problem we solved in Example 7. Using the R vs. F formula, we find

$$R = \frac{\$19,545.76(.015)}{(1.015^{60} - 1)} = \$203.15.$$

That is, I'll pay off my debt to the bank in 60 monthly payments of size \$203.15, beginning one month after the date of the loan.

. . . . *End of Example 8*

In the previous example we used our two previous formulas, one relating F to P, and another relating F to R, to get a relationship between P and R. In general, if we set equal the two expressions for F:

$$P(1+i)^n = \frac{R((1+i)^n - 1)}{i}$$

and solve for P, we get

$$\boxed{P = \frac{R((1+i)^n - 1)}{(1+i)^n i}}$$

If we solve for R instead, the formula becomes

$$\boxed{R = \frac{P(1+i)^n i}{((1+i)^n - 1)}}$$

Example 9 . . . .

You need to borrow some money to buy a car and can afford to pay \$150 per month. If the current NAR on 5-year used car loans is 14.4% compounded monthly, how much can you borrow? What would be the total interest paid on your loan?

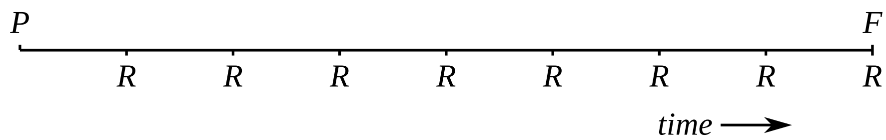
With $n = 60$ monthly payments, monthly interest rate $i = .144 \div 12 = .012$, and monthly payments of size $R = \$150$, the loan amount would be

$$P = \frac{\$150(1.012^{60} - 1)}{1.012^{60} * .012} = \$6389.46.$$

Total payments for this loan would be $60 * \$150 = \$9000$, so the total interest paid must be $\$9000 - \$6389.46 = \$2610.54$. (If you're not keen on paying \$2610.54 in interest charges, borrow less on a shorter term loan!)

. . . . *End of Example 9*

The relationship between the times at which P, F, and R occur can be pictured on a timeline, as in the figure below.



The length of time between P and F is n compounding periods. (In the figure, $n = 8$.) To derive the R-formulas in this handout, it was assumed that the n payments of size R begin one period after the "present" and end on the same day as the "future." This is consistent with practical experience when borrowing money; P dollars are borrowed today to be paid back in n equal payments on size R beginning one interest period after the date of the loan.

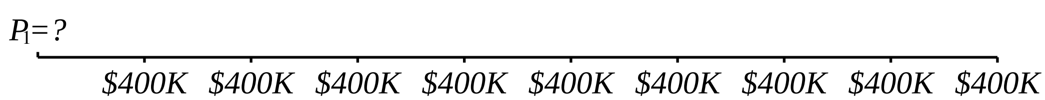
Example 10 . . . .

Suppose the winner of the South Carolina Lottery is to receive four million dollars, paid in 10 yearly installments of \$400,000 each, beginning today. If the current NAR available to the state is 8%, what is the present value of this schedule of payments?

To rephrase the question, how much money would the South Carolina state government have to set aside today to ensure that it can make all 25 payments?

This case is different from the case assumed in our formulas in that the present value is to be calculated for the day of the first payment. In a picture,

We can make do with our existing formulas if we first calculate the present value P_1 of the last 9 payments,



and then add the first payment's value. Using $n = 9$ in the R to P formula, we calculate

$$P_1 = \frac{\$400000((1.08)^9 - 1)}{(1.08)^9(.08)}$$
$$= \$2498755.16,$$

and so the present value of the 10 payments is

$$P = P_1 + \$400000 = \$2898755.16.$$

In effect, the "ten million dollar" prize is actually worth a little under three million today.
. . . . *End of Example 10*

Amortization Schedules

When we pay back a loan with interest in regular payments, it is possible to break each month's payment down into interest and principle, showing exactly how much interest we're paying and what we still owe. (This is also important for tax purposes, since interest paid on home mortgages is currently deductible.) When we do this for every month in the life of the loan, the result is called an **amortization schedule**.

Example 11 . . . .
Suppose you borrow \$1000 at 12% NAR and pay it back in ten months, beginning one month after the day of the loan. Write an amortization schedule for the life of the loan.

The periodic interest rate is .01. Since $P = 1000$, the size R of your monthly payment will be

$$R = \frac{1000(1.01)^{10}.01}{1.01^{10} - 1} = 105.58,$$

so the loan will be paid back in 10 monthly installments of \$105.58.

By the time you make your first payment, you've had the bank's \$1000 for one month, so you owe $.01 * 1000 = 10$ dollars interest. Therefore your first payment reduces your debt to the bank, or **principal**, by \$95.58, and you now owe $\$1000 - \$95.58 = \$904.42$.

month	payment	interest	principal	balance due
0				1000.00
1	105.58	10.00	95.58	904.42

When you make your second payment, you've owed the bank 904.42 for one month, on which the interest due is $.01 * 904.42 = 9.04$, so that much of your payment of 105.58 is interest, and the rest $(105.58 - 9.04 = 96.54)$ reduces the remaining balance of your loan.

month	payment	interest	principal	balance due
0				1000.00
1	105.58	10.00	95.58	904.42
2	105.58	9.04	96.54	807.88

Continuing this way for the life of the loan, we get

month	payment	interest	principal	balance due
0				1000.00
1	105.58	10.00	95.58	904.42
2	105.58	9.04	96.54	807.88
3	105.58	8.08	97.50	710.38
4	105.58	7.10	98.48	611.90
5	105.58	6.12	99.46	512.44
6	105.58	5.12	100.46	411.98
7	105.58	4.12	101.46	310.52
8	105.58	3.11	102.47	208.05

9	105.58	2.08	103.50	104.55
10	105.58	1.05	104.53	0.02

. . . . *End of Example 11*

Note that each month, as your debt decreases, so does the portion of your monthly payment that goes towards interest.

The remaining balance of 2 cents is the result of our having to round all computations to the nearest penny. If we could carry out calculations to an unlimited number of decimals, the final balance would be exactly zero. (In practice, the bank would probably inform the borrower that the final payment should be \$105.60.)

On a loan with longer life, a home mortgage loan for instance, the early payments consist almost entirely of interest, as the next example demonstrates.

Example 12 . . . .
Suppose Junie B. borrows \$200,000 at 6.96% NAR and pays it back over the course of 30 years (typical for home mortgages). Write an amortization schedule for the first year of the loan. How much interest does Junie B. pay in the first year?

Repeating the steps illustrated in Example 11, we arrive at

month	payment	interest	principal	balance due
0				200000.00
1	1325.24	1160.00	165.24	199834.76
2	1325.24	1159.04	166.20	199668.56
3	1325.24	1158.08	167.16	199501.40
4	1325.24	1157.11	168.13	199333.27
5	1325.24	1156.13	169.11	199164.16
6	1325.24	1155.15	170.09	198994.07
7	1325.24	1154.17	171.07	198823.00
8	1325.24	1153.17	172.07	198650.93
9	1325.24	1152.18	173.06	198477.87
10	1325.24	1151.17	174.07	198303.80
11	1325.24	1150.16	175.08	198128.72
12	1325.24	1149.15	176.09	197952.63

By adding the contents of the "interest" column, we find that this borrower paid \$13855.51 in interest during the first year of the loan, a huge sum compared to the \$2047.37 by which the debt was reduced.

. . . . *End of Example 12*

Many mortgages allow the borrower to pay additional principal each month toward toward their balance. Had Junie B. paid just \$100 more per month, that amount would go directly towards reducing the balance:

month	payment	interest	principal	balance due
0				200000.00
1	1425.24	1160.00	265.24	199734.76
2	1425.24	1158.46	266.78	199467.98
3	1425.24	1156.91	268.33	199199.65

4	1425.24	1155.36	269.88	198929.77
5	1425.24	1153.79	271.45	198658.32
6	1425.24	1152.22	273.02	198385.30
7	1425.24	1150.63	274.61	198110.69
8	1425.24	1149.04	276.20	197834.49
9	1425.24	1147.44	277.80	197556.69
10	1425.24	1145.83	279.41	197277.28
11	1425.24	1144.21	281.03	196996.25
12	1425.24	1142.58	282.66	196713.59

Doing this for a year, Junie B. paid \$1200 more, but reduced her debt by \$3286.41; that's \$1239.08 above the total principal \$2047.33 in the earlier table. The extra \$39.08 can be explained by lower interest payments each month, due to lower balances. If she continues to pay the extra \$100 a month she'll pay off her debt more than *five years* ahead of time and save over \$62,000 in interest charges, as we'll see in the next examples.

Example 13 . . . .

If Junie B. in Example 12 continues to pay \$1425.24 per month, how many payments will be necessary to pay off her debt?

To find the required number of payments, we need to find the number n that satisfies the P-to-R equation

$$\$200000 = \$1425.24\frac{1.0058^n - 1}{1.0058^n * .0058}.$$

There's no integer (whole number) n which satisfies this equation. We could find the approximate solution by trial and error, but that could take a *long* time, and we have the tools for finding the answer rather quickly.

To solve for n, first get rid of the fraction by multiplying both sides by the denominator $1.0058^n * .0058$. The result is

$$200000 * .0058 * 1.0058^n = 1425.24(1.0058^n - 1).$$

Simplify $200000 * .0058$ to 1160 and distribue the 1425.24 across the parentheses on the right:

$$1160 * 1.0058^n = 1425.24 * 1.0058^n - 1425.24.$$

Now add 1425.24 to both sides and collect up the 1.0058^n terms:

$$1425.24 = (1425.24 - 1160) * 1.0058^n.$$

Divide by the coefficient of 1.0058^n:

$$\frac{1425.24}{1425.24 - 1160} = 1.0058^n.$$

Take the logarithm of both sides, keeping in mind the rule $\log A^n = n \log A$,

$$\log\left(\frac{1425.24}{1425.24 - 1160}\right) = n \log 1.0058$$

and divide by $\log 1.0058$:

$$n = \frac{\log\left(\frac{1425.24}{1425.24-1160}\right)}{\log 1.0058} = 290.74688\ldots$$

In words, if Junie B. pays \$1425.24 per month, 290 payments will leave her owing money to her lender, and 291 payments will leave the lender owing her. To eliminate the debt exactly, she can make 290 \$1425.24 payments and then a 291th payment of a lesser amount. If she does this, she'll pay off her 30-year mortgage 69 months ahead of schedule.

. . . . *End of Example 13*

The result from the last example can be generalized to a formula for finding n when P, R, and i are known:

$$\boxed{n = \frac{\log\left(\dfrac{R}{R-Pi}\right)}{\log(1+i)}}$$

Example 14 . . . .

If Junie B. pays \$1425.24 per month for 290 months towards her debt, what will be the size of her 291th payment? How much interest will she have paid over the life of her loan?

To find this, we'll first figure what she still owes after making the 290th payment. Compare the future values (on the date of the 290th payment) of \$200000 and 290 payments of size \$1425.24. The \$200000 has future value

$$F = \$200000 * 1.0058^{290} = \$1070047.53$$

while the future value of the 290 payments is

$$F = \frac{\$1425.24(1.0058^{290} - 1)}{.0058} = \$1068988.39$$

After her 290th payment, Junie B.'s debt will be $\$1070047.53 - \$1068988.39 = \$1059.14$. Her 291th payment will occur one month later, by which time her \$1059.14 debt will have grown with interest to

$$\$1059.14(1.0058)^1 = 1065.28.$$

Junie B.'s total payments were $290 * \$1425.24 + \$1065.28 = \$414384.88$. Since her debt was originally \$200000, she paid \$214384.88 in interest. Comparing this with $360 * \$1325.24 - \$200000 = \$277086.40$, we see that paying the extra \$100 a month for 290 months saved her \$62701.52 in interest!

. . . . *End of Example 14*

Note that Junie B.'s last payment of \$1065.28 is approximately her payment size \$1425.24 times the decimal part of n:

$$\$1065.28 \approx \$1425.24 * .74688 = \$1064.48.$$

In other words her owing "290.74688... payments size \$1425.24" means that the 291th partial payment is approximately 74.688% of \$1425.24. The difference between \$1065.28 and \$1064.48 above is not due to round off errors in the calculations. The product of R and the decimal part of n is genuinely different from, but approximately equal to, the true final payment. The nearness of the two is due to a fact from calculus; namely that, when a is any real number and i is close to zero, $\frac{(1+i)^a-1}{i} \approx a$.

Credit Card Debt

Credit card debt differs from, say, a car loan or home mortgage in that the loan is "open-ended." That is, the borrower is under no obligation to pay off the balance in regularly scheduled payments or by a fixed date in time. However, the same rules of compounding interest apply in the long term; interest is computed at some fixed rate i per month, and the borrower's debt rises or falls depending on the size of each month's payment.

(In the short term, for periods of time less than one compounding period, credit card balances and interest-bearing bank accounts accrue interest by the rule of "simple interest," the familiar but otherwise useless "$I = PRT$" formula. This means that when you make purchases on credit or deposit money in your account between the days of the month when your bank computes interest (so that your balance has varied over the interest period), the bank figures interest as i times your *average* daily balance over the interest period.)

Example 15 . . . .
Jimbo Jones has a balance of \$5000 on his EarthBank Visa, which charges 18% NAR compounded monthly. To reduce his debt, Jimbo makes no further purchases with this card and pays a variable amount each month to EarthBank. Suppose he pays \$100, \$175, \$200, and \$180 for the first four months, and then pays \$25 per month (the minimum allowable payment) for the next two months. Find his balance after each payment.

Although Jimbo's loan is not "amortized" (i.e., paid back in equally sized payments) his balance sheet for these six months looks almost exactly like an amortization table, with the exception that the payment differs from month to month. Start with an intial balance of \$5000 and use $i = .18 \div 12 = .015$.

month	payment	interest	principal	balance due
0				5000.00
1	100.00	75.00	25.00	4975.00
2	175.00	74.63	100.37	4874.63
3	200.00	73.12	126.88	4747.75
4	180.00	71.22	108.78	4638.97
5	25.00	69.58	-44.58	4683.55
6	25.00	70.25	-45.25	4728.80

As in our earlier tables, the interest in any row is always i times the previous month's balance, the principal (the amount by which his debt is reduced) is always the payment minus the interest owed, and the balance is always last month's balance minus this month's principal. Notice that, since the interest due depends on last month's balance and is independent of the size of the payment, the higher the payment, the greater the principal. Jimbo was reducing his debt until months 5 and 6, when his payment wasn't enough to cover the interest owed those months. In this case, the principal is negative, and his debt rose, by $44.58 in month 5 and by $45.25 in month 6.

. . . . *End of Example 15*

It is not unusual for a credit card company to require a "minimum payment" that's less than the interest owed that month, so that the borower's debt rises. This is a way for the card company to increase the size of its loan to the cardholdrer; ultimately, this means more interest paid to the lender, which, after all, is how the lender makes its money.

Example 16 . . . .

Jimbo Jones from Example 15 decides to pay $200 per month toward his debt on his EarthBank Visa. What will be his debts after 25 payments?

Although we could complete an balance sheet for the next 25 months, and find the answer in the 25th row, it's easier to compare the future values of $5000 and of 25 payments of size $250.

The future value of the $5000 is

$$F = \$5000 * 1.015^{25} = \$7254.73$$

while the future value of the 25 payments is

$$F = \frac{\$200(1.015^{25} - 1)}{.015} = \$6012.60.$$

After his 25th payment, Jimbo's debt will be $\$7254.73 - \$6012.60 = \$1242.13$. Even though his total payments were $25 * \$200 = \$5000$, his debt was reduced only by $3757.87 due to the interest charged to his account.

. . . . *End of Example 16*

Example 17 . . . .

How many payments of $200 per month would it take for Jimbo to pay off his $5000 debt?

Using the "$n = \ldots$" formula earlier, we find that when $P = 5000$ and $R = 200$,

$$n = \frac{\log\left(\dfrac{200}{200 - 5000 * .015}\right)}{\log(1.015)} = 31.56799\ldots$$

This means that, if Jimbo pays $200 per month, 31 payments will leave him owing money to his card company, and 32 payments will leave the company owing him. To eliminate his debt exactly, he can make 31 $200 payments and then a 32nd payment of approximately $\$200 * .56799 = \$113.60$.

. . . . *End of Example 17*

Problems

With these formulas, we can solve a wide variety of problems involving interest. A good first step is to identify which of the variables P, F, R, N, i you've given and which of these you're being asked to find. That by itself will almost finish the simpler problems. Remember that the only variables which stand for dollar amounts are P, F, and R. Always do your calculations with i expressed in decimal form, e.g., $25\% = 0.25$.

1. If \$5000 is invested in an account paying 12% NAR compounded monthly, what will be the balance in the account after 4 years? After 8 years? After 12 years? 16 years?
2. \$100 is deposited in an account paying 5% NAR compounded semiannually. After 3 years the balance is transferred to an account paying 6% NAR compounded quarterly. What is the balance 7 years after making the transfer?
3. What amount, deposited today at 7.08% NAR compounded monthly will result in \$10,000 after 15 years?
4. I anticipate needing \$10,000 in 7 years for a down payment on a house. If the best savings account I can find pays a effective annual yield of 11%, how much must I deposit today to reach my goal?
5. I wish to save \$10,000 in 7 years by making monthly deposits into an account that pays 9% NAR compounded monthly. How much must I deposit?
6. Bank A pays its depositors an NAR of 5.1% compounded quarterly, while Bank B pays NAR 5.04% compounded monthly. Calculate the effective annual yield at each bank. Which bank gives a better deal?
7. If you have a choice between saving your money in one account that pays NAR 15% compounded semiannually and another paying NAR 12% compounded daily, which should you choose?
8. A bank advertises that it pays 5.0625% effective annual yield. If it compounds interest twice a year, what NAR is it using? (Hint: $(1+i)^2 = 1.050625$.)
9. A bank advertises that it pays 5.25% effective annual rate. If it compounds interest once a month, what NAR is it using in its calculations?
10. One share of qwerty.com's stock was worth \$22.00 on Jan 1, 1998. On Jan 1, 2002, the same share was worth \$54.50. What was the stock's average annual rate of growth?
11. One share of yuiop.com's stock was worth \$22.00 on Jan 1, 1995. On Jan 1, 1998, the same share was worth only \$15.00. What was the stock's average annual rate of decline?
12. If a home in Charleston was worth \$50,000 in 1987, and the same home was worth \$225,000 in 2001, what was the average annual rate of growth of the value of the house?
13. Verify the factoring rules by multiplying out the left side.
    a. $(x-1)(x^4+x^3+x^2+x+1) = x^5-1$
    b. $(x-1)(x^5+x^4+x^3+x^2+x+1) = x^6-1$

14. Find the sums
    a. $2 + 2 * 1.05 + 2 * 1.05^2 + \cdots + 2 * 1.05^{100} = ?$
    b. $2 * 1.05^{100} + 2 * 1.05^{101} + \cdots + 2 * 1.05^{200} = ?$
15. Find the sums
    a. $2 + 2 * 2.1 + 2 * 2.1^2 + \cdots + 2 * 2.1^{99} + 2 * 2.1^{100} = ?$
    c. $2 - 2 * 2.1 + 2 * 2.1^2 - \cdots - 2 * 2.1^{99} + 2 * 2.1^{100} = ?$
16. If I save \$150 each quarter year in an account that pays 4% NAR compounded quarterly, how much will I have accumulated after 40 payments?
17. If I invest \$100 each month in an account paying 5.4% NAR compounded monthly, how much will I have accumulated in ten years?
18. How much must I deposit today in an account paying NAR 7% compounded annually in order to make 50 yearly withdrawals of \$1000, beginning in one year?
19. How much must I deposit today in an account paying 6% NAR compounded annually in order to make 40 yearly withdrawals of \$750, beginning in one year?
20. To ensure an income for your aged mother, you wish to deposit money today in an account paying 9% NAR compounded monthly and allow her to make monthly withdrawals of \$150 for the next ten years, beginning next month. How much must you deposit today to achieve this?
21. You are about to deposit a check in your mother's account that will give her the income described in Problem 20, when she tells you that she won't need the the money for another 3 years. How much should you deposit today to ensure her an income of \$150 for ten years, beginning 3 years later than originally planned?
22. Your mother's needs are the same as explained in Problem 21, but you decide to save for her income by monthly deposits instead of one lump sum. How much should you deposit in her account each month for the next three years to guarantee her \$150 per month for ten years?
23. At the birth of your first child, you decide to start saving for her college education. How much should you deposit each month in an account paying 3.96% NAR compounded monthly in order to save \$50,000 in 18 years?
24. How much must I deposit today in an account paying 6% NAR compounded quarterly in order to make 25 yearly withdrawals of \$1000, beginning one year from the initial deposit?
25. If I deposit \$120.00 a month for 60 months into an account that pays 4.8% NAR compounded monthly, how much money is in my account two years after my last deposit?
26. Suppose I deposit \$4000 into an account that pays 4.8% NAR compounded monthly. Five years later, I decide to withdraw the entire balance in 24 equal monthly withdrawls, begining in one month. What should be the size of my withdrawls?
27. A car buyer borrows \$10,000 at NAR 10.2% compounded monthly, and begins paying off the two-year loan one month later. Complete an amortization schedule for the first five months of the loan.

28. If you borrow \$7000 at 12% NAR compounded monthly, what must be your monthly payments to pay off the loan
    a. in 1 year? In 5 years? In 25 years? In 50 years?
    b. Could the monthly payment ever be below \$70? Explain.
29. The State Lottery pays its winners 10 million dollars in 20 yearly installments of \$500,000, beginning today. If the current NAR available is 12% compounded annually, what is the present value of such a schedule of payments? (Or, what price could you fetch for a winning ticket?)
30. If I borrow \$5000 at 6% NAR compounded monthly and pay it back over 24 months, how much interest will I pay during the first 3 months? How much interest will I pay over the life of the loan? (Hint: it is not necessary to complete a 24 month amortization schedule to answer this second part.)
31. Suppose that you borrowed \$10,000 at 8.7% NAR compounded monthly to be repaid over ten years. After paying for five years, you now wish to increase your monthly payments so that the loan will be paid off in two more years. What are the sizes of your monthly payments during the first five years and the last two years? (Hint: at any time, the remaining balance on your loan is the present value of the remaining payments.)
32. A homeowner is paying \$1000 a month on a 30-year mortgage at 7.2% NAR compounded monthly. He has 240 payments to go. What does he still owe?
33. A college student has \$20,000 in student loans at 6.99% NAR compounded monthly to be paid back over 10 years. After making payments for eight years, the former student comes into some money and wishes to pay off the remaining balance on the loan. What lump sum payment will pay off the loan two years early? (Hint: you do not need to complete an 8 year amortization schedule to answer this question.)
34. If the student in Problem 33 pays off her loan two years earlier as suggested, how much interest will she save? That is, how much interest will she have paid over the eight years of the loan, and how much interest would she have paid had she paid the money back in ten years as originally planned?
35. A firsttime car buyer borrows \$7000 on a one-year loan at 6% NAR compounded monthly. After three payments, the buyer realizes that he cannot afford to pay off the loan so quickly, so he contacts the bank, which agrees to let him pay off the remaining balance in two years, still at 6% compounded monthly. What was the size of his first three payments, and what was the size of his last 24 payments?
36. See the last example. If Jimbo Jones makes 31 montly payments of size \$200 to his credit card company, what should be the exact size of the 32nd payment in order to eliminate his debt? Hint: what will be his debt immediately after making the 31st payment?

37. A homeowner is paying \$1000 a month on a 30-year mortgage at 7.2% NAR compounded monthly. He has 240 payments to go and still owes \$127008.43, when he decides that he can now afford to pay an extra \$500 each month towards this debt. How many \$1500 payments will it take to pay off his debt, and what will be the approximate size of this last (partial) payment?

38. About how much interest did the homeowner in Problem 37 save by increasing his payments to \$1500? (Use your answer in Problem 37 when figuring his total payments.)

39. A consumer currently has a balance of \$6500 on a credit card that charges interest at 12% NAR compounded monthly. Write a balance sheet for this loan for the next five months if the payments are \$100, \$200, \$120, \$50, and \$75.

40. A consumer with a balance of \$7500 on a credit card that charges 12.6% NAR compounded monthly wishes to pay off the debt entirely in 36 payments, beginning next month. What should be the size of her payments?

41. A consumer with a balance of \$7500 on a credit card that charges 12.6% NAR compounded monthly wishes to reduce her debt to \$1000 in 36 payments, beginning next month. What should be the size of her payments so that her balance will be \$1000 after the 36th payment? (Hint: What is the present value of that \$1000? How much of her original debt must she pay off?)

42. If I have \$3200 debt on a credit card that charges me 18% NAR compounded monthly, and I pay \$100 a month, how many months will it take for me to pay off my debt?

Selected Answers

1. \$8061.13 in 4 yrs; \$12,996.36 in 8 yrs; \$20,953.08 in 12 yrs. 2. \$175.95 3. \$3,468.44 4. \$4,816,58 5. \$85.89 6. A pays 5.198%; B pays 5.158%. (A is better.) 8. 5% compounded semiannually. 9. approximately 5.1277533% 10. approx. 25.46% 11. approx. −11.98% 12. approx. 11.34% 14. a. 5483.05... b. 721028.34... 16. \$7,332.96 17. \$15,865.10 18. \$13,800.75 19. \$11,284.72 20. \$11841.25 21. \$9048.48 22. \$287.74 23. \$159.07 24. \$12,619.39 25. \$8935.59 26. \$222.52 27. int., prin., and bal. for 1st 5 months are 85.00, 377.37, 9622.63; 81.79, 380.58, 9242.05; 78.56, 383.82, 8858.23; 75.29, 387.08, 8471.15; 72.00, 390.37, 8080.78; 29. \$4,182,888.43 32. \$127008.43 33. \$5184.81 35. Originally, $R = \$602.47$. After 3 payments, \$5289.09 still owed. Last 24 payments are each \$234.43. 36. \$113.96 (answer to hint is \$112.28) 37. 118; 119th payment will be approximately \$867.75. 38. \$863.87; \$868.85; tot. int. in last 20 yrs of original loan would be \$112991.57; tot. int. under new plan will be \$50859.32, a savings of \$62132.25. 39. payment, int., prin., and bal. for months 1-5 are: \$100, \$65, \$35, \$6465; \$200, \$64.65, \$135.35, \$6329.65; \$120, \$63.30, \$56.70, \$6272.95; \$50, \$62.73, \$-12.73, \$6285.68; \$75, \$62.86, \$12.14, \$6273.54; 41. \$228.26 (answer to hint is \$686.58, \$6813.42) 42. 44. (final payment will be approximately \$92.12.)