More on Favard interpolation from subsets of a rectangular lattice

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Abstract

The problem of Favard interpolation from subsets of a rectangular lattice is addressed, paying particular attention to two desirable properties of the interpolant: localness and a bound independent of the aspect ratio of the lattice. We see circumstances in which these are mutually exclusive and constructions that offer one but not the other.

Key Words and Phrases: interpolation, divided differences, multivariate.

1. Introduction

An interpolation theorem of Favard [1] states that, if f is a real-valued function on an increasing sequence of real numbers $(m_i)_{i\in I}$ and if n is a positive integer, then f has a smooth extension F whose nth derivative is bounded by f's nth divided differences:

$$|D^n F| \le C(n) \max_i |\mathbf{\Delta}(m_i, \ldots, m_{i+n}) f|.$$

That is, f has an extension whose nth derivative is not more than a constant times its necessary minimum size. Favard's extension F depends locally and linearly on f, and the constant $C(n)$ is independent of both f and the data points $(m_i)_{i\in I}$. (See [8, Thm. 3.1] for the proof of this and more in case $(m_i)_I$ is bi-infinite, i.e., $I = \mathbb{Z}$. Favard's original result for finite I follows easily.)

This result is the motivation for what could be called the Favard Interpolation Problem, or **FIP** for short: to find an extension of function values (or **data**) on a discrete set of points (or **data sites**) in $\mathbb{R}^k$ so that, for some particular n, the nth derivatives of the extension are no more than a constant times some nth multivariate divided differences of the data, which here we take to mean f's tensor product divided differences of total order n. These are a natural choice for their recurrence relations and their dependence solely on function values at points, but they limit the data sets we can consider to subsets of the Cartesian product of k sequences of real numbers. As it turns out [8, Thm. 4.5], if $n > 1$, these sequences must be arithmetic for the FIP to have a solution, effectively forcing the data set to lie on a lattice $M\mathbb{Z}^k$ for some $k \times k$ diagonal matrix M.

(See [6, 7] for results on multivariate divided differences that consist entirely of function evaluations.)

This paper is a continuation of earlier work [9] on a specific formulation of the FIP (Problem 3.1 below) dealing with the case that the function f is defined only on some proper subset of $M\mathbb{Z}^k$. It's a multivariate analogue of Favard's original problem in case the data set is finite, but while the finite case follows easily from the infinite case in one variable, the same is not true in several variables. The FIP is solvable on every lattice [5] but may or may not have a solution on a subset of the same lattice, depending on the geometry of that set [9].

Previously [9, Thm. 5.5], it was discovered that when the data set satisfies a simple geometric condition (Condition 4.15 below), Problem 3.1 has a local and linear solution for which the constant in (3.3) is independent of the data set. This paper deals with what happens when this condition is not met, and the results fall into two parts.

First, when the data set fails to meet Condition 4.15, we can't rule out a local solution to the FIP, but we can prove (Section 4) that there's a limit to *how* local any solution can be. If the constant in (3.3) is to be independent of M (in which case we say that the solution is **scale-independent**), then localness is limited even more. On some sets which violate Condition 4.15, the FIP has no scale-independent solution which deserves to be called local.

Second, we give two constructions that produce a local but scale-dependent solution on some data sets (Section 5), and a scale-independent but nonlocal (or less local) solution on others (Section 6), including, in both cases, sets on which such solutions are known via results in Section 4 to be the only ones possible.

The multivariate Favard interpolation theorem from [5] has been used by Holtby [2, 3] to arrive at bounds on solutions to multivariate difference equations and by Preston [10] in the study of solutions to partial differential equations modeling the motion of whips and chains. Holtby's work also relies on a result from [4] that has only been published (in significantly extended form) as [9, Thm. 5.5].

We begin by establishing some notation in Section 2 and give a precise statement of the problem under consideration in Section 3.

2. Notation

The ith component of a point x in $\mathbb{R}^k$ is denoted $x(i)$. If M is a $k \times k$ matrix, then the image of $x \in \mathbb{R}^k$ under M is the product Mx of M and (the column vector) x. If X is a subset of $\mathbb{R}^k$, then

$$MX := \{Mx : x \in X\}.$$

If M is a positive diagonal matrix, its aspect ratio is

$$r(M) := \max_{i,j} M(i,i)/M(j,j).$$

For x and y in $\mathbb{R}^k$, we say $x \leq y$ if $x(i) \leq y(i)$ for all i and $x < y$ if $x(i) < y(i)$ for all i. Note that $x \leq y$ and $x \neq y$ do not imply $x < y$. Let

$$[x, y] := \{u \in \mathbb{R}^k : x \leq u \leq y\},$$

The elements of $\mathbb{Z}^k$ are called **multiintegers**, and the multiinteger whose every component is 1 is denoted

$$\mathbb{1} := (1, 1, \ldots, 1).$$

In case x and y are in $\mathbb{Z}^k$, let

$$\{x, \ldots, y\} := \mathbb{Z}^k \cap [x, y].$$

If z is a multiinteger, then the set $[z, z + \mathbb{1}]$ is called a **cell**. For instance, a cell in $\mathbb{R}^3$ is a closed cube of volume one with multiinteger vertices.

We'll use the usual $\pm$ notation for addition of sets, or of sets and vectors. For instance, $z+[0, \mathbb{1}] = [z, z+\mathbb{1}]$. The set $A-B$ should not be confused with $A \setminus B$, i.e., the intersection of A and $B^c :=$ the complement of B.

If Ω is a set in $\mathbb{R}^k$, define

$$\mathbb{Z}_\Omega := \Omega \cap \mathbb{Z}^k,$$

the set of all multiintegers in Ω, and abbreviate the $L_\infty(\Omega)$ norm as

$$\| \cdot \|_\Omega := \| \cdot \|_{L_\infty(\Omega)}.$$

The vector e_i is the ith column of the $k \times k$ identity matrix. The set of **multiindices**, i.e., those multiintegers with nonnegative components, is written $\mathbb{Z}^k_+$. If α is a multiindex, then $\alpha! := \prod_i (\alpha(i)!)$, and $|\alpha| := \sum_i \alpha(i)$. The αth power function is denoted

$$()^\alpha : x \in \mathbb{R}^k \mapsto x^\alpha := \prod_{i=1}^{k} x(i)^{\alpha(i)}.$$

The gradient operator is denoted

$$D := \left[\frac{\partial}{\partial x(1)}, \frac{\partial}{\partial x(2)}, \cdots, \frac{\partial}{\partial x(k)} \right],$$

so that, naturally,

$$D^\alpha := \left(\frac{\partial}{\partial x(1)} \right)^{\alpha(1)} \cdots \left(\frac{\partial}{\partial x(k)} \right)^{\alpha(k)}.$$

Let

$$\mathbf{\Delta}(x_0, \ldots, x_n) f$$

denote the nth divided difference of f at the real numbers $x_0, \ldots, x_n$, with the usual meaning if some $x_i = x_j$. When $z \in \mathbb{Z}^k$ and $\alpha \in \mathbb{Z}^k_+$, define

$$\diamondsuit_z^\alpha := \bigotimes_{i=1}^{k} \mathbf{\Delta}(z(i), z(i) + 1, \ldots, z(i) + \alpha(i)),$$

that is, the tensor product divided difference that acts on k-variate functions by applying $\mathbf{\Delta}(z(i), z(i) + 1, \ldots, z(i) + \alpha(i))$ in the ith variable for each $i = 1, \ldots, k$. For any positive diagonal matrix M, define

$$\diamondsuit_{M,z}^\alpha : f \mapsto \operatorname{diag}(M)^{-\alpha} \diamondsuit_z^\alpha \big(f \circ M \big).$$

The **total order** of D^α and of $\diamondsuit_z^\alpha$ is $|\alpha|$. Define the polynomial spaces

$$\Pi_{<n} := \operatorname{span}\{ ()^\alpha : |\alpha| < n \}$$

and

$$\Pi_{<n\mathbb{1}} := \operatorname{span}\{ ()^\alpha : \forall i \, \alpha(i) < n \}.$$

3. The problem and an immediate consequence

Problem 3.1. *Let n and k be positive integers, and let Ω be a connected union of cells in $\mathbb{R}^k$ with the property that no nonzero polynomial in $\Pi_{<n}$ is identically zero on $\mathbb{Z}_\Omega$ (i.e., $\mathbb{Z}_\Omega$ is* **total** *for $\Pi_{<n}$). Let M be a positive diagonal matrix. Find an operator $F_{\Omega,M}$ mapping functions f defined on $M\mathbb{Z}_\Omega$ to functions $F_{\Omega,M}f$ possessing all derivatives of total order n on $M\Omega$ so that*

$$F_{\Omega,M}f = f \text{ on } M\mathbb{Z}_\Omega \tag{3.2}$$

and

$$\max_{|\alpha|=n} \|D^\alpha F_{\Omega,M}f\|_{L_\infty(M\Omega)} \leq C \max\{\|\Diamond^\alpha_{M,z}f\| : |\alpha| = n,\ [z, z+\alpha] \subset \Omega\} \tag{3.3}$$

for some constant C independent of f.

The condition that $\mathbb{Z}_\Omega$ (and therefore $M\mathbb{Z}_\Omega$) is total for $\Pi_{<n}$ does not appear in the problem as stated in [9] but leads to some conclusions regarding any solution $F_{\Omega,M}$ and its localness. For instance, (3.3) implies that $F_{\Omega,M}$ maps $\Pi_{<n}$ into itself, and, since nothing in $\Pi_{<n} \setminus \{0\}$ is zero on $M\mathbb{Z}_\Omega$, $F_{\Omega,M}$ must reproduce $\Pi_{<n}$:

Lemma 3.4. *If $F_{\Omega,M}$ solves Problem 3.1 and if $f \in \Pi_{<n}$, then $F_{\Omega,M}f = f$.*

For instance, $F_{\Omega,M}0 = 0$ even if $F_{\Omega,M}$ is nonlinear.

4. Local and scale-independent solutions

A solution $F_{\Omega,M}$ to Problem 3.1 is called **scale-independent** when the constant C in (3.3) is independent of M and **local** when both of the following conditions are true. First, for every cell $U = [u, u + \mathbb{1}]$ in Ω and every positive diagonal matrix M, there exists a set

$$\sigma_{U,M} \subset \mathbb{Z}_\Omega$$

such that, for any function $f : M\mathbb{Z}_\Omega \to \mathbb{R}$, the restriction of $F_{\Omega,M}f$ to the set MU depends entirely on f's values on $M\sigma_{U,M}$. Second, the set of multiintegers $\sigma_{U,M} - u$ is bounded independently of U and M.

In Theorem 4.10 and the examples that follow, we see that, for some sets Ω, there are limits to how small the bound on $\sigma_{U,M} - u$ can be if $F_{\Omega,M}$ is to be both scale-independent and local.

We begin with some elementary observations about $\sigma_{U,M}$.

Lemma 4.1. *The following are true whenever $F_{\Omega,M}$ is a local solution to Problem 3.1 and $U = [u, u + \mathbb{1}]$ is a cell in Ω.*

(4.2) $\{u, \ldots, u + \mathbb{1}\} \subset \sigma_{U,M}$.

(4.3) *If $f = p$ on $M\sigma_{U,M}$ for some $p \in \Pi_{<n}$, then $F_{\Omega,M}f|_{MU} = p|_{MU}$.*

(4.4) *No nonzero $p \in \Pi_{<n}$ is identically zero on $\sigma_{U,M}$.*

(4.5) $\#\{z(i) : z \in \sigma_{U,M}\} \geq n$ *for every* $i \in \{1, \ldots, k\}$.

(4.6) *The diameter of* $\sigma_{U,M}$ *in every coordinate direction is* $\geq n - 1$.

Proof: (4.2): Because $F_{\Omega,M} f = f$ on $M\mathbb{Z}_\Omega$, to change f on $M\{u, \ldots, u+\mathbb{1}\}$ is to change the values of $F_{\Omega,M} f$ at points in $M[u, u + \mathbb{1}]$.

(4.3): Since $f = p$ on $M\sigma_{U,M}$, the restrictions of $F_{\Omega,M} f$ and of $F_{\Omega,M} p$ to MU must be the same. By Lemma 3.4, $F_{\Omega,M} p = p$.

(4.4): If $p \in \Pi_{<n}$ is identically zero on $\sigma_{U,M}$, then $p \circ M^{-1} \in \Pi_{<n}$ is identically zero on $M\sigma_{U,M}$, and therefore $F_{\Omega,M}(p \circ M^{-1}) = F_{\Omega,M} 0 = 0$ on MU. By Lemma 3.4, $p \circ M^{-1} = 0$ on MU, and therefore p is identically zero.

(4.5): Suppose that, for some i, the number of distinct elements in $A := \{z(i) : z \in \sigma_{U,M}\}$ is less than n. Then the nonzero polynomial $p(x) := \prod_{c \in A} \big(x(i) - c\big)$ is 0 on $\sigma_{U,M}$, contradicting (4.4).

(4.6): If the diameter of $\sigma_{U,M}$ in some coordinate direction e_i is $< n - 1$, then $\#\{z(i) : z \in \sigma_{U,M}\} < n$, contradicting (4.5). □

Example 4.7: Let Ω be the two-toned shaded subset of $\mathbb{R}^2$ in Figure 4.8, and let the points marked $\bullet$ be the multiinteger points $\mathbb{Z}_\Omega$ of Ω. No nonzero polynomial of degree less than 3 vanishes on $\mathbb{Z}_\Omega$, so Ω satisfies the hypotheses of Problem 3.1 with $n = 3$. Let U be the cell filled with the darker shade of gray.

By (4.6), if there is a local solution $F_{\Omega,M}$ to Problem 3.1 on Ω, then for each positive diagonal matrix M, the set $\sigma_{U,M}$ cannot be contained in the points marked /. That is, the restriction of $F_{\Omega,I} f$ to U cannot depend solely on the values of f at the bottom two rows of multiintegers. □

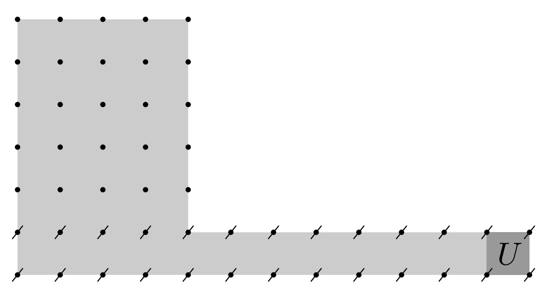

Figure 4.8 ($n = 3$)

Even more can be said about $\sigma_{U,M}$ in case $F_{\Omega,M}$ is scale-independent. For instance, as will be seen in Example 4.16, if Ω and $\mathbb{Z}_\Omega$ are the sets pictured in Figure 4.9, and if $F_{\Omega,M}$ is a local and scale-independent solution to Problem 3.1, then, as $r(M) \to \infty$, either $\sigma_{U,M}$ must include a point marked only 1 (i.e., not also marked 0) or $\sigma_{V,M}$ must include a point marked only 0. The next theorem is a generalization of this example.

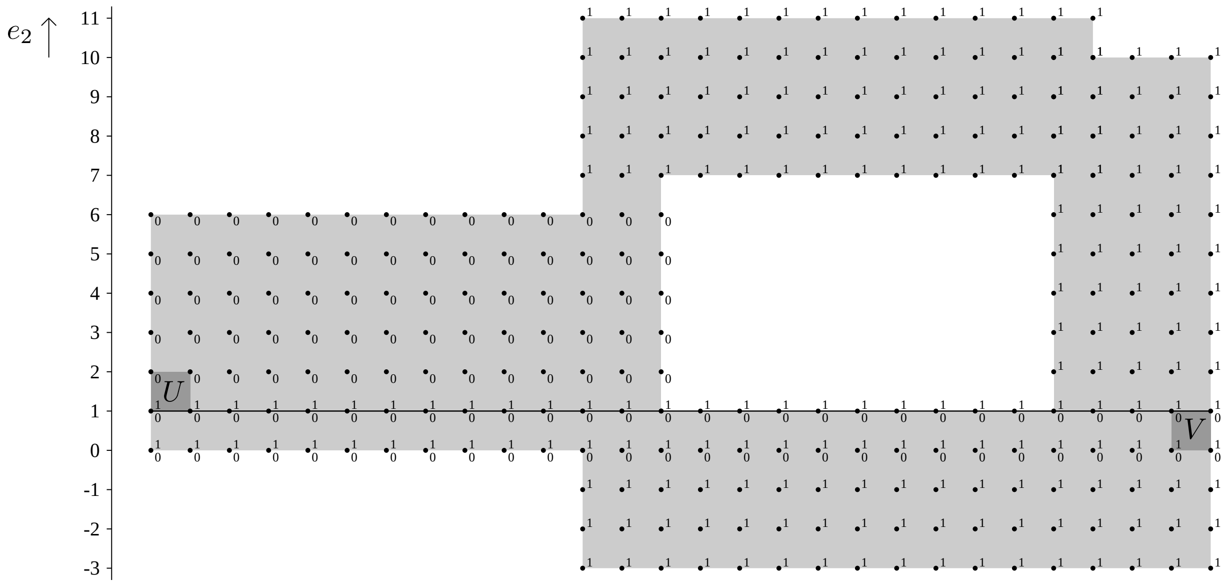

Figure 4.9 ($n = 3$)

A subset ξ of $\mathbb{Z}_\Omega \times \{0,1\}$ is called a **multivalued function** from $\mathbb{Z}_\Omega$ into $\{0,1\}$ if, for each z in $\mathbb{Z}_\Omega$, there is a point in ξ whose first coordinate is z. When the point (z,i) belongs to ξ, we write $\xi(z) = i$. The set $\{z \in \mathbb{Z}_\Omega : \xi(z) = i\}$ is denoted $\xi^{-1}(i)$. Note that $\xi^{-1}(1)$ and $\xi^{-1}(0)$ are not necessarily disjoint.

Theorem 4.10. *Let n, k, and Ω be as in Problem 3.1. Choose integers i, j, and c so that $i \in \{1,\ldots,k\}$ and $c \in \{j,\ldots,j+n-2\}$. Choose a set $\Lambda \subset \{e_1,\ldots,e_k\} \setminus \{e_i\}$.*

Let E be a connected component of $\{x \in \Omega : x(i) = c\}$. Let U and V be cells in Ω so that there exists a path L in E from U to V traveling only in directions in span(Λ).

Let

$$\mathbb{Z}_\Omega^{i,j} := \{z \in \mathbb{Z}_\Omega : z(i) \in \{j,\ldots,j+n-2\}\},$$

and let ξ be a multivalued function from $\mathbb{Z}_\Omega$ into $\{0,1\}$ satisfying the following three conditions.

(4.11) $\xi^{-1}(0) \cap \xi^{-1}(1) = \mathbb{Z}_\Omega^{i,j}$.

(That is, ξ is multivalued exactly on $\mathbb{Z}_\Omega^{i,j}$.)

(4.12) *$\xi^{-1}(1) \setminus \xi^{-1}(0)$ is finite.*

(That is, ξ is 1 at only finitely many points outside of $\mathbb{Z}_\Omega^{i,j}$.)

(4.13) *If $e_m \in \Lambda$ and if both z and $z + e_m$ are in $\mathbb{Z}_\Omega \setminus \mathbb{Z}_\Omega^{i,j}$ and the line segment $[z, z+e_m]$ lies in Ω, then $\xi(z) = \xi(z+e_m)$.*

(That is, where it is single valued, ξ is constant at the multiintegers along lines in Ω in the directions of Λ.)

Suppose also that Problem 3.1 has a local and scale-independent solution $F_{\Omega,M}$ and that the nth derivatives of $F_{\Omega,M} f$ are integrable for all functions f on $M\mathbb{Z}_\Omega$. Then it is impossible for both $\sigma_{U,M}$ to remain a subset of $\xi^{-1}(0)$ and $\sigma_{V,M}$ to remain a subset of $\xi^{-1}(1)$ as $r(M) \to \infty$.

One could always construct a multivalued function ξ to satisfy (4.11), (4.12), and (4.13) by defining ξ equal both 0 and 1 on $\mathbb{Z}_\Omega^{i,j}$ and 0 elsewhere (although in that case the conclusion of Theorem 4.10 is trivial).

Proof: Define the function

$$g := \xi p,$$

where

$$p(x) := \big(x(i) - j\big)\big(x(i) - j - 1\big) \cdots \big(x(i) - j - n + 2\big).$$

Note that $p = 0$ on $\mathbb{Z}_\Omega^{i,j}$, so, by (4.11), g is single-valued on $\mathbb{Z}_\Omega$. Since g is supported on the finite set $\xi^{-1}(1) \setminus \xi^{-1}(0)$, the tensor product divided differences

$$\{\Diamond_z^\alpha g : |\alpha| = n, [z, z + \alpha] \subset \Omega\} \tag{4.14}$$

are bounded. What's more, the only nonzero members of (4.14) are those for which $\alpha \in \operatorname{span}(\Lambda^c)$, since p and ξ are constant in the directions Λ.

For any positive diagonal matrix M, define the functions

$$p_M := p \circ M^{-1} \qquad g_M := g \circ M^{-1}.$$

If $\sigma_{V,M} \subset \xi^{-1}(1)$ for some positive diagonal matrix M, then $g_M = p_M$ on $M\sigma_{V,M}$ and so, by (4.3), $F_{\Omega,M} g_M = p_M$ on MV. Similarly, if $\sigma_{U,M} \subset \xi^{-1}(0)$ for some M, then $g_M = 0$ on $M\sigma_{U,M}$ and $F_{\Omega,M} g_M = 0$ on MU.

For any number $\varepsilon > 0$, define M_ε to be the diagonal matrix given by the rule

$$M_\varepsilon(\ell, \ell) := \begin{cases} \varepsilon & \text{if } e_\ell \in \Lambda, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

The nonzero members of

$$\left\{\Diamond_{M_\varepsilon, z}^\alpha g_{M_\varepsilon} : |\alpha| = n, [z, z + \alpha] \subset \Omega\right\}$$

are exactly the same as those in (4.14) and thus are bounded independently of ε, say by a constant K.

Let a and b be the endpoints in U and V, respectively, of the path L. If both $\sigma_{U,M_\varepsilon} \subset \xi^{-1}(0)$ and $\sigma_{V,M_\varepsilon} \subset \xi^{-1}(1)$ for all $\varepsilon > 0$, then the path integral of the gradient

$$\int_{M_\varepsilon L} \left(D D_{e_i}^{n-1} F_{\Omega,M_\varepsilon} g_{M_\varepsilon}(x)\right)^T dx = D_{e_i}^{n-1} F_{\Omega,M_\varepsilon} g_{M_\varepsilon}\big|_{M_\varepsilon a}^{M_\varepsilon b} = (n-1)!.$$

On the other hand, this integral is bounded above by the length of the path $M_\varepsilon L$ times the maximum of the nth derivatives of $F_{\Omega,M_\varepsilon} g_{M_\varepsilon}$ on $M_\varepsilon \Omega$ times some constant $B(n, k)$:

$$(n-1)! \leq \varepsilon \operatorname{length}(L) B(n, k) \max_{|\alpha| = n} \|D^\alpha F_{\Omega,M_\varepsilon} g_{M_\varepsilon}\|_{M_\varepsilon \Omega}$$

so that, by (3.3),

$$(n-1)! \leq \varepsilon \operatorname{length}(L) B(n, k) \cdot C \cdot K,$$

where, by hypothesis, C is independent of ε. Letting $\varepsilon \to 0$ arrives at a contradiction and completes the proof. □

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Earlier [9, Thm. 5.5], it was seen that Problem 3.1 has a local, linear, scale-independent solution when Ω satisfies the following geometric condition [9, Cond. 5.1].

Condition 4.15. *For every integer i in $\{1,\dots,k\}$ and every integer c and every connected component E of $\{x \in \Omega : x(i) = c\}$, there exists an integer j_E in $\{c-n+1,\dots,c\}$ so that, for every point x in E,*

$$x + (j_E - x(i))e_i + e_i[0, n-1] \subset \Omega.$$

(That is, the closed line segment between the two points

$$\big(x(1),\dots,\overset{j_E}{\cancel{x(i)}},\dots,x(k)\big) \quad\text{and}\quad \big(x(1),\dots,\overset{j_E+n-1}{\cancel{x(i)}},\dots,x(k)\big)$$

is contained entirely in Ω.)

To put it loosely, a set Ω violates Condition 4.15 when it is too narrow in the direction e_i along some E. To apply Theorem 4.10 to such an Ω, choose cells U and V intersecting E as far apart as possible, choose directions $\Lambda \not\ni e_i$ spanning a path in E from U to V, define $\xi^{-1}(1) \cap \xi^{-1}(0)$ to be some $\mathbb{Z}_\Omega^{i,j}$ containing the multiinteger points of E, and assign the remaining points in $\mathbb{Z}_\Omega$ to either $\xi^{-1}(1)$ or $\xi^{-1}(0)$ so as to satisfy (4.12) and (4.13). One could trivially do this by choosing $U = V$ and defining $\xi = 0$ on $\mathbb{Z}_\Omega \setminus \mathbb{Z}_\Omega^{i,j}$, but the strength of the conclusion of Theorem 4.10 depends on how far one can make U from $\xi^{-1}(1)\setminus\xi^{-1}(0)$ and, at the same time, V from $\xi^{-1}(0)\setminus\xi^{-1}(1)$. In some instances, including Examples 4.7 and 4.16, we can see that [9, Thm. 5.5] fails to produce a local, scale-independent solution because, practically speaking, no such solution exists.

Example 4.16: For instance, let $k = 2$, let Ω be the shaded subset of $\mathbb{R}^2$ in Figure 4.9, and let the points marked $\bullet$ be the multiinteger points $\mathbb{Z}_\Omega$.

Let $n = 3$, $i = 2$, and $c = 1$, so that E in Condition 4.15 is the line segment —— along which $x(2) = 1$ in Ω . Because there's no integer j_E in $\{-1, 0, 1\}$ for which $x + (j_E - 1)e_2 + e_2[0, 2] \subset \Omega$ for all $x \in E$, the set Ω violates Condition 4.15, and [9, Thm. 5.5] fails to produce a local, scale-independent solution to Problem 3.1 on Ω.

To apply Theorem 4.10, let $j = 0$ and $\Lambda = \{e_1\}$. Pick U and V to be the cells as marked in the figure, and let ξ be the multivalued function whose values at each point $\bullet$ in $\mathbb{Z}_\Omega$ are marked.

Line (4.11) is true, since $\xi =$ both 0 and 1 only on $\mathbb{Z}_\Omega^{i,j} = \mathbb{Z}_\Omega^{2,0}$, the points z in $\mathbb{Z}_\Omega$ at which $z(2) = 0$ or 1. Line (4.12) is trivially true, since $\mathbb{Z}_\Omega$ is finite, and line (4.13) is true, since ξ is constant at the elements of $\mathbb{Z}_\Omega \setminus \mathbb{Z}_\Omega^{2,0}$ lying on lines in Ω parallel to e_1. Theorem 4.10 states that, as $r(M) \to \infty$, it is impossible for both $\sigma_{U,M} \subset \xi^{-1}(0)$ and $\sigma_{V,M} \subset \xi^{-1}(1)$. That is, if $F_{\Omega,M}$ is scale-independent, then as $r(M) \to \infty$, either $F_{\Omega,M}f|_{MU}$ must depend on f at some point not marked 0 or $F_{\Omega,M}f|_{MV}$ must depend on f at some point not marked 1. □

Example 4.17: The set Ω in Figure 4.18 violates Condition 4.15 when $k = i = 2$, $c = 0$, and $n = 3$, but here the conclusion of Theorem 4.10 is much weaker than in the previous example. The constancy of ξ along (horizontal) lines parallel to e_1 as required by (4.13) makes it impossible for any U and V to both be very far from $\xi^{-1}(1)\setminus\xi^{-1}(0)$ and

$\xi^{-1}(0) \setminus \xi^{-1}(1)$, respectively. The requirement that the interpolant on U must depend on some data points not marked 0, or that its values on V must depend on some data points not marked 1 would hardly make a solution nonlocal in a practical sense. (Just such a solution is constructed in Example 6.12.) □

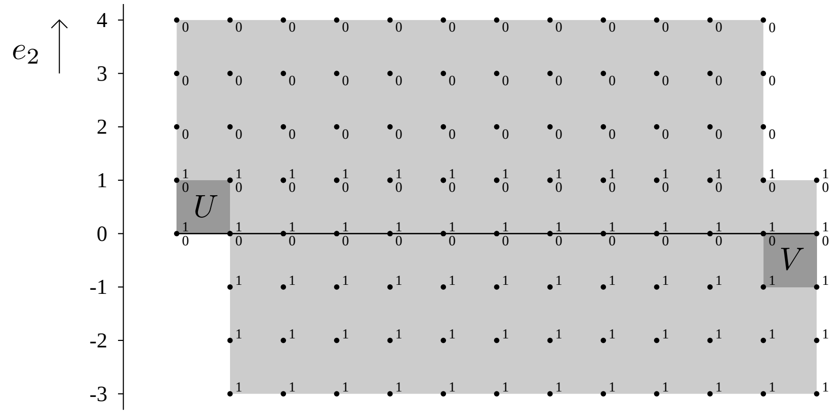

Figure 4.18 ($n = 3$)

5. A local but scale-dependent solution

In this section and the next, we discuss Problem 3.1 on sets violating Condition 4.15. Under some additional assumptions on Ω, we produce interpolants that are the best possible in light of Theorem 4.10: one that is local but scale-dependent (Theorem 5.3) and one that is scale-independent but potentially nonlocal (Theorem 6.7). In both of these, and unlike the interpolant produced in [9, Thm. 5.5], the constant in (3.3) depends on Ω, but it does so in a quantifiable way. In (5.8), it depends on the integer K_Ω in Condition 5.1, while in (6.11) it depends on the map $f \mapsto f^+$, which in turn depends on how many times one must apply Theorem 6.1 to go from Ω to Ω^+.

If $w \in \mathbb{Z}^k$, let

$$[\![w]\!] := [w, w + (n-1)\mathbb{1}]$$

and

$$R_w : \mathbb{R}^{\mathbb{Z}_{[\![w]\!]}} \to \Pi_{<n\mathbb{1}}$$

be the linear operator that maps a function f defined on $\mathbb{Z}_{[\![w]\!]}$ to its unique interpolant $R_w f$ in $\Pi_{<n\mathbb{1}}$.

Condition 5.1. *There exists a mapping $w : \mathbb{Z}_\Omega \to \mathbb{Z}_\Omega : u \mapsto w_u$ and a positive integer K_Ω with the following two properties. First, $u \in [\![w_u]\!] \subset \Omega$ for each $u \in \mathbb{Z}_\Omega$. Second, if u and v are in $\mathbb{Z}_\Omega$ and are both contained in some cell in Ω, then there exists a sequence*

$$w_u = p_0, p_1, p_2, \ldots, p_\ell = w_v$$

of at most $1 + K_\Omega$ points in $\mathbb{Z}_\Omega$ such that each $[\![p_i]\!] \subset \Omega$ and every two adjacent terms p_{i-1} and p_i differ by exactly one in exactly one coordinate. (That is, $\forall i \exists j$ so that $p_i - p_{i-1} = \pm e_j$.)

Example 5.2: If $n = 4$ and Ω is the two-toned region in Figure 5.20 whose multiinteger points $\mathbb{Z}_\Omega$ are marked $\bullet$, then Ω satisfies Condition 5.1 with $K_\Omega = 19$. In fact, Ω is itself the union of the 20 sets $[p_i, p_i + 3\mathbb{1}]$ for the multiintegers marked $p_0, \ldots, p_{19}$ in that figure.

Theorem 5.3. *Let n, k, and Ω be as in Problem 3.1, and suppose that Ω satisfies Condition 5.1. Then there exists a mapping G_Ω from the set of all positive diagonal matrices M and functions $f : M\mathbb{Z}_\Omega \to \mathbb{R}$ to functions $G_{\Omega,M}f$ with the following properties:*
(5.4) $G_{\Omega,M}f \in C^\infty(M\Omega)$.
(5.5) $G_{\Omega,M}f = f$ *on* $M\mathbb{Z}_\Omega$.
(5.6) $G_{\Omega,M}f$ *depends linearly on* f.
(5.7) $G_{\Omega,M}f$ *depends locally on* f. *Specifically,* $\sigma_{U,M} \subset \{u - (n-1)\mathbb{1}, \ldots, u + n\mathbb{1}\}$ *for any cell* $U = [u, u + \mathbb{1}] \subset \Omega$.
(5.8) *There exists a constant* $C(n, k, K_\Omega)$ *so that, for any cell* $U = [u, u + \mathbb{1}]$ *in* Ω,

$$\max\{\|D^\alpha G_{\Omega,M}f\|_{MU} : |\alpha| = n\} \le C(k, n, K_\Omega) r(M)^n \max\nolimits_0 |\diamondsuit^\alpha_{M,z} f|$$

where $\max_0$ *is taken over all* $z \in \mathbb{Z}^k$ *and* $\alpha \in \mathbb{Z}^k_+$ *for which* $|\alpha| = n$, $[z, z + \alpha] \subset \Omega$ *and* $\|z - u\|_\infty \le K_\Omega + 2n - 2$.

Proof: It will suffice to prove Theorem 5.3 in case M is the $k \times k$ identity matrix, since the general result then follows for

$$G_{\Omega,M}f := \Big(G_{\Omega,I}\big(f \circ M)\big)\Big) \circ M^{-1}. \tag{5.9}$$

Assume that M is the identity for the remainder of the proof.

Choose an infinitely differentiable function $\Psi : \mathbb{R}^k \to \mathbb{R}$ for which

$$\operatorname{supp}\Psi \subset [-\mathbb{1}, \mathbb{1}] \quad \text{and} \quad \sum_{v \in \mathbb{Z}^k} \Psi(\cdot - v) = 1. \tag{5.10}$$

Then, for any u and v in $\mathbb{Z}^k$,

$$\Psi(u - v) = \begin{cases} 1 & \text{if } u = v, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

Furthermore, on $[u, u + \mathbb{1}]$, the only nonzero terms of the sum in (5.10) correspond to $v \in \{u, \ldots, u + \mathbb{1}\}$, so

$$\sum_{v \in \{u, \ldots, u + \mathbb{1}\}} \Psi(\cdot - v) = 1 \quad \text{on} \quad [u, u + \mathbb{1}]. \tag{5.12}$$

Define the operator

$$G_{\Omega,I} := \sum_{v \in \mathbb{Z}_\Omega} \Psi(\cdot - v) R_{w_v}$$

and suppose that $f : \mathbb{Z}_\Omega \to \mathbb{R}$. By Condition 5.1, the interpolation points for each R_{w_v} lie in $\mathbb{Z}_\Omega$, so $G_{\Omega,I}f$ is well defined. (5.4) is immediate.

Since each R_{w_v} is linear, so is $G_{\Omega,I}$, proving (5.6). What's more, $G_{\Omega,I}$ is local: on $[u, u+\mathbb{1}]$, the values of $G_{\Omega,I}f$ are determined by the values of f at the interpolation points of the operators $\{R_{w_v} : u \le v \le u+\mathbb{1}\}$. These points are at most $n-1$ units from v in each coordinate and therefore lie in $\{u-(n-1)\mathbb{1}, \ldots, u+n\mathbb{1}\}$, proving (5.7).

If $u \in \mathbb{Z}_\Omega$, then

$$G_{\Omega,I}f(u) = \sum_{v\in\mathbb{Z}_\Omega} \Psi(u-v)R_{w_v}f(v) = R_{w_u}f(u),$$

by (5.11), and this equals $f(u)$, since u is one of the interpolation points of the operator R_{w_u}. This proves (5.5).

Next, fix an $\alpha \in \mathbb{Z}_+^k$ for which $|\alpha| = n$ and a cell $[u, u+\mathbb{1}] \subset \Omega$. Equation (5.12) implies that, on $[u, u+\mathbb{1}]$,

$$G_{\Omega,I}f = R_{w_u}f + \sum_{v\in\{u,\ldots,u+\mathbb{1}\}} \Psi(\cdot - v)(R_{w_v} - R_{w_u})f. \tag{5.13}$$

To prove (5.8), it will suffice to bound the αth derivatives of $R_{w_u}f$ and of $\Psi(\cdot - v)(R_{w_v} - R_{w_u})f$ by some of the divided differences found in the right side of (5.8) times a constant depending on n, k, α, and K_Ω.

Write $R_{w_u}f$ as

$$R_{w_u}f = \sum_{0\le\beta<\mathbb{1}} \diamondsuit_{w_u}^{\beta} f$$

where N_β is the polynomial

$$N_\beta : \mathbb{R}^k \to \mathbb{R} : x \mapsto \prod_{\substack{1\le i\le k\\ 0\le j<\beta(i)}} \big(x(i) - j\big).$$

The αth derivative of N_β is zero unless $\alpha \le \beta$, so

$$D^\alpha R_{w_u}f = \sum_{\alpha\le\beta<n\mathbb{1}} D^\alpha N_\beta(\cdot - w_u)\diamondsuit_{w_u}^{\beta} f.$$

Therefore

$$\|D^\alpha R_{w_u}f\|_{[u,u+\mathbb{1}]} \le \sum_{\alpha\le\beta<n\mathbb{1}} \|D^\alpha N_\beta(\cdot + u - w_u)\|_{[0,\mathbb{1}]} |\diamondsuit_{w_u}^{\beta} f|. \tag{5.14}$$

It is not hard to show that, for any multiindices α and γ and multiinteger z,

$$\diamondsuit_z^\gamma \diamondsuit_\cdot^\alpha = \binom{\gamma+\alpha}{\alpha} \diamondsuit_z^{\gamma+\alpha}, \tag{5.15}$$

where

$$\binom{\gamma+\alpha}{\alpha} := \frac{(\gamma+\alpha)!}{\alpha!\gamma!}.$$

So, each tensor product divided difference $\diamondsuit^{\beta}_{w_u}$ in (5.14) is a linear combination of

$$\{\diamondsuit^{\alpha}_{z} : w_u \le z \le w_u + \beta - \alpha\}$$

with coefficients that can bounded by some constant $C_{0,n,k}$ depending on n and k. In addition, $[z, z+\alpha] \subset \Omega$ for each such $\diamondsuit^{\alpha}_{z}$, because $[\![w_u]\!] \subset \Omega$ by Condition 5.1. Consequently,

$$\|D^{\alpha} R_{w_u} f\|_{[u,u+\mathbb{1}]} \le C_{1,n,k} \max\nolimits_1 \|D^{\alpha} N_{\beta}(\cdot + t)\|_{[0,\mathbb{1}]} \max\nolimits_2 |\diamondsuit^{\alpha}_{z} f|$$

where $\max_1$ is taken over all β and t in $\{0, \dots, (n-1)\mathbb{1}\}$, and $\max_2$ is taken over all z in $\{u-(n-1)\mathbb{1}, \dots, u+(n-1)\mathbb{1}\}$ for which $[z, z+\alpha] \subset \Omega$, and $C_{1,n,k}$ is a constant depending on n and k.

Next, we'll bound the αth derivative of the sum in (5.13).

For any $v \in \{u, \dots, u + \mathbb{1}\}$, let

$$w_u = p_0, p_1, \dots, p_{\ell} = w_v$$

be the sequence of points guaranteed by Condition 5.1 and rewrite the term in (5.13) as

$$(R_{w_v} - R_{w_u})f = \sum_{j=1}^{\ell} (R_{p_j} - R_{p_{j-1}})f.$$

For each j in this sum, p_j and p_{j-1} differ by one in exactly one coordinate, say the ith. Let p denote the point with the smaller ith coordinate. Then $q := p + e_i$ is the other.

The polynomial $(R_q - R_p)f$ has degree $< n$ in each variable. As such, it can be written

$$(R_q - R_p)f = \sum_{0 \le \beta < n\mathbb{1}} N_{\beta}(\cdot - q) \diamondsuit^{\beta}_{q} (R_q - R_p)f. \tag{5.16}$$

Since $R_q f$ and $R_p f$ agree at those interpolation points common to both R_q and R_p, $\diamondsuit^{\beta}_{q}(R_q - R_p)f = 0$ unless $\beta(i) = n - 1$. For any such β,

$$\diamondsuit^{\beta}_{q} R_q f = \diamondsuit^{\beta}_{q} f$$

(since $R_q f = f$ on $\{q, \dots, q + \beta\}$) and

$$\diamondsuit^{\beta}_{q} R_p f = \sum_{\beta \le \gamma < n\mathbb{1}} \diamondsuit^{\beta}_{q} N_{\gamma}(\cdot - p) \diamondsuit^{\gamma}_{p} f.$$

Any γ in this sum must satisfy $\gamma(i) = n - 1$, since $\beta(i) = n - 1$. Since $p = q$ in all components except the ith,

$$\diamondsuit^{\beta}_{q} N_{\gamma}(\cdot - p) = \diamondsuit^{\beta}_{p} N_{\gamma}(\cdot - p) = \begin{cases} 1 & \text{if } \beta = \gamma, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\lozenge_q^\beta R_p f = \lozenge_p^\beta f$, and (5.16) becomes

$$\begin{aligned}(R_q - R_p)f &= \sum_{\substack{0\le\beta<n\mathbb{1}\\ \beta(i)=n-1}} N_\beta(\cdot - q)(\lozenge_q^\beta - \lozenge_p^\beta)f\\ &= \sum_{\substack{0\le\beta<n\mathbb{1}\\ \beta(i)=n-1}} N_\beta(\cdot - q)n\lozenge_p^{\beta+e_i} f.\end{aligned}$$

Therefore

$$D^\alpha\big(\Psi(\cdot - v)(R_q - R_p)f\big) = \sum_{\substack{0\le\beta<n\mathbb{1}\\ \beta(i)=n-1}} D^\alpha\big(\Psi(\cdot - v)N_\beta(\cdot - q)\big)n\lozenge_p^{\beta+e_i} f,$$

and

$$\begin{aligned}&\|D^\alpha\big(\Psi(\cdot - v)(R_q - R_p)f\big)\|_{[u,u+\mathbb{1}]}\\ &\qquad\le \sum_{\substack{0\le\beta<n\mathbb{1}\\ \beta(i)=n-1}} \|D^\alpha\big(\Psi(\cdot-(v-u))N_\beta(\cdot + u - q)\big)\|_{[0,\mathbb{1}]}n|\lozenge_p^{\beta+e_i} f|.\end{aligned}$$

By (5.15), each $\lozenge_p^{\beta+e_i}$ above is a linear combination of

$$\{\lozenge_z^{ne_i} : p \le z \le p + \beta - (n-1)e_i\}$$

with coefficients that can be bounded above by some constant $C_{2,n,k}$ depending on n and k. Any such z must lie in $\{u-(n-1+K_\Omega)\mathbb{1},\dots,u+(2n-2+K_\Omega)\mathbb{1}\}$, because $\beta \le (n-1)\mathbb{1}$ and

$$\|u - p\|_\infty \le \|u - w_u\|_\infty + \|w_u - p\|_\infty \le n - 1 + K_\Omega. \tag{5.17}$$

Furthermore, $[z, z + ne_i] \subset [\![p]\!] \cup [\![q]\!]$, which lies in Ω by Condition 5.1. By the argument in (5.17) again, $\|u - q\|_\infty \le n - 1 + K_\Omega$. Consequently,

$$\begin{aligned}&\big\|D^\alpha\big(\Psi(\cdot - v)(R_q - R_p)f\big\|_{[u,u+\mathbb{1}]}\\ &\qquad\le C_{3,n,k}\max\nolimits_3\|D^\alpha\big(\Psi(\cdot - t)N_\beta(\cdot + s)\big)\|_{[0,\mathbb{1}]}\max\nolimits_4|\lozenge_z^{ne_i} f|,\end{aligned} \tag{5.18}$$

where, $\max_3$ is taken over all multiindices $\beta < n\mathbb{1}$, all $t \in \{0, \mathbb{1}\}$, and all

$$s \in \{-(n-1+K_\Omega)\mathbb{1},\dots,(n-1+K_\Omega)\mathbb{1}\},$$

$\max_4$ is taken over all $z \in \{u-(n-1+K_\Omega)\mathbb{1},\dots,u+(2n-2+K_\Omega)\mathbb{1}\}$ for which $[z, z + ne_i] \subset \Omega$, and $C_{3,n,k}$ is a constant depending on n and k. Therefore

$$\Big\|D^\alpha \sum_{u\le v\le u+\mathbb{1}} \Psi(\cdot - v)(R_{w_v} - R_{w_u})f\Big\|_{[u,u+\mathbb{1}]}$$

is less or equal the right side of (5.18) times $2^k K_\Omega$. This competes the proof of (5.8) and of Theorem 5.3. □

It may seem paradoxical that, to bound the derivatives of $G_{\Omega,I}f$ on U, the right side of (5.8) must include tensor product divided differences not supported on $\sigma_{U,M}$. However, such a situation must occur if, as in the next example, these functionals are needed to span some locally supported n**th difference**, i.e., a linear combination of point evaluations that vanishes on $\Pi_{<n}$ [9, Def. 3.4].

Example 5.19: Suppose that $n = 4$ and that Ω is the two-toned region in Figure 5.20 whose multiinteger points $\mathbb{Z}_\Omega$ are marked $\bullet$. Let Θ be the subset of Ω that is filled with the darker shade of gray and whose multiinteger points $\mathbb{Z}_\Theta$ are also marked $\bigcirc$.

As seen in Example 5.2, Ω satisfies Condition 5.1 with $K_\Omega = 19$. Theorem 5.3 therefore guarantees that Problem 3.1 has a solution, albeit a scale-dependent one, on Ω.

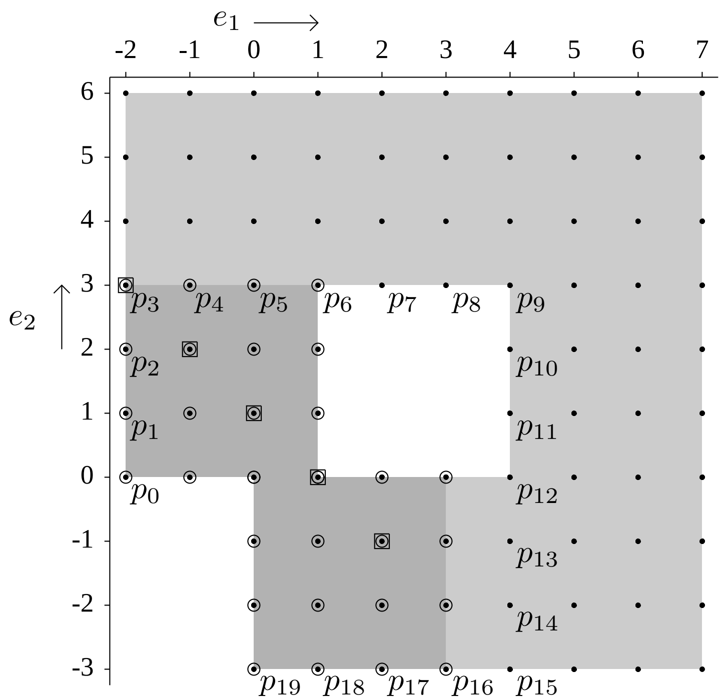

Figure 5.20 ($n = 4$)

By choosing w_u to be p_0 for all $u \in \{(-2,0),\ldots,(1,3)\}$ and p_{19} for all other u in $\mathbb{Z}_\Theta$, one can construct $G_{\Omega,I}$ so that the restriction of $G_{\Omega,I}f$ to Θ depends solely on f's values on $\mathbb{Z}_\Theta$. However, the fourth total order derivatives of $G_{\Omega,I}f$ or, for that matter, of any interpolant to f, cannot be bounded by the fourth total order tensor product divided differences of f supported on $\mathbb{Z}_\Theta$. That is, Problem 3.1 has no solution on Θ.

To see this, observe that the function

$$f(x) := \begin{cases} 0 & \text{if } x(2) \geq 0 \\ x(1)\big(x(1)-1\big)x(2) & \text{if } x(2) < 0 \end{cases}$$

is annihilated by each of the fourth total order tensor product divided differences supported on Z_Θ but that the fourth divided difference of f in the direction $e_1 - e_2$ at the points marked □ is nonzero. Consequently, any smooth interpolant to f on $\mathbb{Z}_\Theta$ must have a nonzero fourth derivative somewhere along the line from $(-2,3)$ to $(2,-1)$. (See also [9, Ex. 3.11].) □

6. A scale-independent but nonlocal solution

In this last section, we address Problem 3.1 once more on sets violating Condition 4.15. By Theorem 4.10, we can't expect solutions to be both scale-independent and local, but under an additional assumption on Ω, we construct a solution (Theorem 6.7) that is scale-independent and nonlocal, or at least less local than in [9, Thm. 5.5]. The main idea is to extend $f : \mathbb{Z}_\Omega \to \mathbb{R}$ to a larger domain, one point at a time, without increasing the nth divided differences of f too much (Theorem 6.1), until Ω has been replaced by a larger set satisfying Condition 4.15.

First, we make two definitions needed for the statement of the next result.

If $x \in \mathbb{R}^k$, define $\mathbb{1}_x$ to be the characteristic function of the support of x:

$$\mathbb{1}_x(j) := \begin{cases} 1 & \text{if } x(j) \neq 0, \text{ and} \\ 0 & \text{if } x(j) = 0. \end{cases}$$

If a and b are points in $\mathbb{R}^k$ for which $a \le b$, define the set

$$(a,b] := \left\{ x \in \mathbb{R}^k : \begin{array}{ll} a(j) < x(j) \le b(j) & \text{if } a(j) < b(j), \text{ and} \\ x(j) = a(j) & \text{if } a(j) = b(j). \end{array} \right\}$$

Theorem 6.1. *Let n, k, and Ω be as in Problem 3.1 and let $p \in \mathbb{Z}^k \setminus \mathbb{Z}_\Omega$. Suppose that there exists a nonzero multiindex $\beta \le n\mathbb{1}$ satisfying*

(6.2) $\quad \forall \gamma \in \{0,\ldots,\beta\} \setminus \{\beta\},\ [p-\beta, p-\beta+\gamma] \subset \Omega$

and that, if α is a nonzero multiindex $\le n\mathbb{1}$ satisfying $[p-\alpha, p-\alpha+\gamma] \subset \Omega$ for all $\gamma \in \{0,\ldots,\alpha\} \setminus \{\alpha\}$, then $\alpha \le \beta$. (That is, among all multiindices satisfying (6.2), β is maximal.)

Let

$$\Omega^+ := \Omega \cup (p - \mathbb{1}_\beta, p].$$

Then for every $f : \mathbb{Z}_\Omega \to \mathbb{R}$ there is an extension $f^+ : \mathbb{Z}_{\Omega^+} \to \mathbb{R}$ with the following properties.

(6.3) $\quad f^+ = f$ *on* $\mathbb{Z}_\Omega$.

(6.4) $\quad f \mapsto f^+$ *is a linear map.*

(6.5) $\quad$ *For some constant $C(n,k)$ and for any multiindex $\alpha \le n\mathbb{1}$,*

$$\max\{|\Diamond_z^\alpha f^+| : [z, z+\alpha] \subset \Omega^+\} \le C(n,k) \max\{|\Diamond_z^\alpha f| : [z, z+\alpha] \subset \Omega\}.$$

Proof: The only multiinteger in $(p - \mathbb{1}_\beta, p]$ is p itself, so to construct f^+ satisfying (6.3), it is only necessary to define $f^+(p)$.

By (6.2), the multiintegers $\{p-\beta,\ldots,p-\beta+\gamma\}$ lie in Ω for each γ in $\{0\ldots,\beta\}\setminus\{\beta\}$, so $\{p-\beta,\ldots,p\}\setminus\{p\}\subset\mathbb{Z}_\Omega$. Choose $f^+(p)$ so as to make $\diamondsuit^\beta_{p-\beta}f^+=0$. Note that $f^+(p)$ depends linearly on f's values on $\{p-\beta,\ldots,p\}\setminus\{p\}$, with coefficients that can be bounded in terms of n and k, proving both (6.4) and the special case of (6.5) when $\alpha=0$.

To prove (6.5) in general, take $z\in\mathbb{Z}^k$ and $\alpha\in\mathbb{Z}^k_+\setminus\{0\}$ so that $[z,z+\alpha]$ is a subset of Ω^+ but not of Ω. Then $[z,z+\alpha]$ must have nonempty intersection with $(p-\mathbb{1}_\beta,p]$. That is, some x in $\mathbb{R}^k$ lies in $[z,z+\alpha]\cap(p-\mathbb{1}_\beta,p]$. If $i\in\operatorname{supp}\beta$, then $p(i)-1<x(i)\le(z+\alpha)(i)$, which implies that $p(i)\le(z+\alpha)(i)$, and if $i\notin\operatorname{supp}\beta$, then $p(i)=x(i)\le(z+\alpha)(i)$. Consequently, $p\le z+\alpha$.

Now suppose that $p(j)<(z+\alpha)(j)$ for some j. Since both $p(j)$ and $(z+\alpha)(j)$ are integers, $p(j)+1/2$ must also be less than $(z+\alpha)(j)$, and therefore $p+1/2e_j\in[z,z+\alpha]\subset\Omega^+$. Since it does not lie in $(p-\mathbb{1}_\beta,p]$, the point $p+1/2e_j$ must lie in Ω, and because Ω is a union of closed cells, $p+1/2e_j\in[u,u+\mathbb{1}]\subset\Omega$ for some multiinteger u . That is, $u(j)\le p(j)+1/2\le u(j)+1$, implying $p(j)=u(j)$, and $u(i)\le p(i)\le u(i)+1$ for all i other than j, implying that $p(i)$ is one of $u(i)$ or $u(i)+1$. Consequently, p is one of the multiintegers in $[u,u+\mathbb{1}]$, contradicting the hypothesis that $p\notin\mathbb{Z}_\Omega$. Therefore, p must equal $z+\alpha$.

We'll next see that $\alpha\le\beta$. Suppose that $\gamma\in\{0,\ldots,\alpha\}\setminus\{\alpha\}$ and $x\in[p-\alpha,p-\alpha+\gamma]=[z,z+\gamma]$. Then $z\le x\le z+\gamma\le z+\alpha=p$, so $x\in[z,z+\alpha]$. Because $\gamma\ne\alpha$, there's an integer j for which $(z+\gamma)(j)\le p(j)-1$, which implies that x cannot be in $(p-\mathbb{1}_\beta,p]$. That is,

$$[z,z+\gamma]\subset[z,z+\alpha]\setminus(p-\mathbb{1}_\beta,p]\subset\Omega.$$

By hypothesis, this implies $\alpha\le\beta$.

Because $\diamondsuit^\beta_{p-\beta}f^+=0$, and because $\diamondsuit^\beta_{p-\beta}$ is a linear combination of

$$\bigl\{\diamondsuit^\alpha_u : u\in\{p-\beta,\ldots,z\}\bigr\}$$

with coefficients that can be bounded in terms of n and k, $\diamondsuit^\alpha_z f^+$ is a linear combination of

$$\bigl\{\diamondsuit^\alpha_u f : u\in\{p-\beta,\ldots,z\}\setminus\{z\}\bigr\} \tag{6.6}$$

with coefficients that can be similarly bounded. For any u as in (6.6), $[u,u+\alpha]\subset[p-\beta,p]$ and $u+\alpha\ne p$ together imply the existence of an integer j for which $(u+\alpha)(j)\le p(j)-1$. Such j must lie in the support of β, and so

$$[u,u+\alpha]\subset[p-\beta,p-\beta+(\beta-e_j)]\subset\Omega$$

by (6.2). As a result,

$$|\diamondsuit^\alpha_z f^+|\le C(n,k)\max\{|\diamondsuit^\alpha_u f| : [u,u+\alpha]\subset\Omega\},$$

completing the proof of (6.5). □

Theorem 6.7. *Let n and k be positive integers, and let Ω and Ω^+ be sets as in Problem 3.1. Suppose that $\Omega \subset \Omega^+$, that Ω^+ satisfies Condition 4.15, and that there exists a linear map $f \mapsto f^+$ from functions on $\mathbb{Z}_\Omega$ to functions on $\mathbb{Z}_{\Omega^+}$ such that f^+ agrees with f on $\mathbb{Z}_\Omega$, and $\forall \alpha \le n\mathbb{1}$,*

$$\max\{|\Diamond_z^\alpha f^+| : [z, z+\alpha] \subset \Omega^+\} \le C(n,k,\Omega) \max\{|\Diamond_z^\alpha f| : [z, z+\alpha] \subset \Omega\}.$$

Then there exists a mapping H_Ω from the set of all positive diagonal matrices M and functions $f : M\mathbb{Z}_\Omega \to \mathbb{R}$ to functions $H_{\Omega,M} f$ with the following properties:
(6.8) $H_{\Omega,M} f \in C^\infty(M\Omega)$.
(6.9) $H_{\Omega,M} f = f$ *on* $M\mathbb{Z}_\Omega$.
(6.10) $H_{\Omega,M} f$ *depends linearly on* f.
(6.11) *There exists a constant $C(n,k,\Omega)$ so that, for any cell $[u, u+\mathbb{1}] \subset \Omega$ and for any $\alpha \le n\mathbb{1}$,*

$$\|D^\alpha H_{\Omega,M} f\|_{M[u,u+\mathbb{1}]} \le C(n,k,\Omega) \max\{|\Diamond_{M,z}^\alpha f| : [z, z+\alpha] \subset \Omega\}.$$

Proof: Let $H_{\Omega,I} : f \mapsto (F_{\Omega^+,I} f^+)|_\Omega$, where F is the operator constructed in [9, Thm. 5.5], and then obtain $H_{\Omega,M}$ from $H_{\Omega,I}$ by scaling, as in (5.9). □

Example 6.12: It was noted in Example 4.17 that the set Ω, seen again in Figure 6.13, might admit a local, scale-independent solution to Problem 3.1 when $n = 3$. In fact, Theorem 6.7 does just that, after an application of Theorem 6.1 (with $\beta = 3\mathbb{1}$) appends the point p to $\mathbb{Z}_\Omega$ and the dashed cell to Ω to create a set Ω^+ which satisfies Condition 4.15.

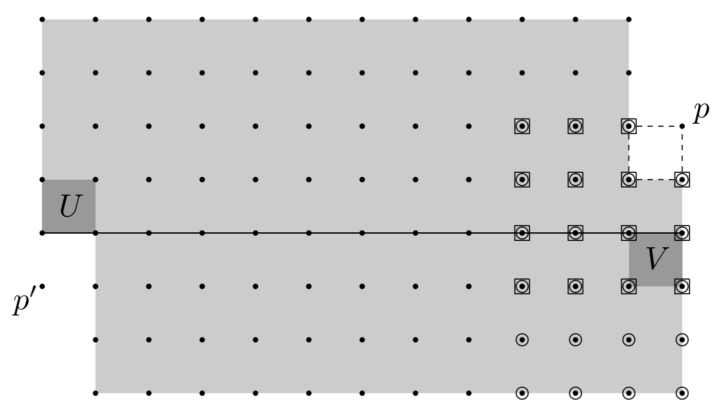

Figure 6.13 ($n = 3$)

We know, by [9, Thm. 5.5], that the values of $F_{\Omega^+,I} f^+$ on any cell $[u, u+\mathbb{1}] \subset \Omega^+$ depend only on the values of f^+ at the points $\{u - 2\mathbb{1}, \ldots, u + 3\mathbb{1}\} \cap \mathbb{Z}_{\Omega^+}$, and, by Theorem 6.1, that $f^+(p)$ depends on f's values at the points $\{p - \beta, \ldots, p\} \setminus \{p\}$ (marked □). So, the values of $H_{\Omega,I} f$ on V depend only on the values of f at the points in $\mathbb{Z}_\Omega$ marked ◯.

This is consistent with Example 4.17, where it was shown that in any scale-independent solution to Problem 3.1 on Ω, either the interpolant's values on V must depend in part on data at points not marked 1, or its values on U must depend in part on data at points not marked 0, as would have been the case had we instead appended the point p' to $\mathbb{Z}_\Omega$ by applying Theorem 6.1 to $-\Omega$. $\square$

As noted in the last example, one limitation of Theorem 6.1, that it only appends to $\mathbb{Z}_\Omega$ points in the positive directions (e.g., up and to the right in $\mathbb{R}^2$), can be overcome by reorienting Ω. The details are in the lemma below, which we state without proof.

Lemma 6.14. *Let N be a diagonal matrix of 1's and -1's. Suppose the sets $\Omega \subset \Omega^+$ have the property that each function f defined on $\mathbb{Z}_\Omega$ has an extension f^+ to $\mathbb{Z}_{\Omega^+}$ satisfying (6.3), (6.4), and (6.5). Then the same must be true of $N\Omega$ and $N\Omega^+$.*

Example 6.15: Suppose that $n = 3$ and Ω is the set in Figure 6.16. Condition 4.15 is violated along the horizontal line. If, by seven applications of Theorem 6.1, we extend f to the points $1, 2, \ldots, 7$ and apply Theorem 6.7, then the values of $H_{\Omega,I} f$ on the darker square would depend on $f^+(7)$ and therefore on f's values on the points marked $\square$ (as well as those marked $\bigcirc$).

Though scale-independent, $H_{\Omega,I}$ is not local in a practical sense (although it is as local as possible, which can be seen by the methods used in Example 4.17). $\square$

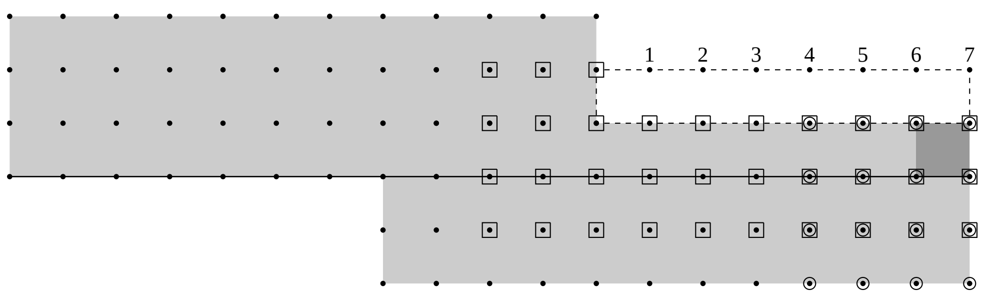

Figure 6.16 ($n = 3$)

We end with an example of when this method fails and some speculation in general about why failure occurs.

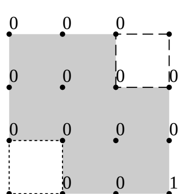

Figure 6.17 ($n = 3$)

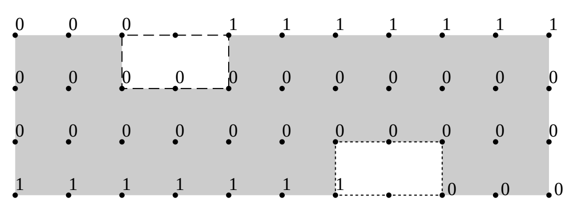

Figure 6.18 ($n = 3$)

Example 6.19:

If Ω is the shaded set in Figure 6.17, then enlarging Ω to a Ω^+ that satisfies Condition 4.15 would require appending one of the two dashed regions to Ω. Theorem 6.1 fails to accomplish this. For example, if p were the additional point in the upper right corner of the figure, then both $(3,2)$ and $(2,3)$ satisfy (6.2), but there's no maximal β satisfying (6.2).

In fact, no such extension scheme $f \mapsto f^+$ is possible. If we let $f = 0$ or 1 on $\mathbb{Z}_\Omega$ as illustrated, then, regardless of which of the dashed regions we add to Ω, there's no extension f^+ that will satisfy both

$$\max\{|\diamondsuit_z^{(3,0)} f^+| : [z, z+3e_1] \subset \mathbb{Z}_{\Omega^+}\} \le C \max\{|\diamondsuit_z^{(3,0)} f| : [z, z+3e_1] \subset \mathbb{Z}_\Omega\}$$

and

$$\max\{|\diamondsuit_z^{(0,3)} f^+| : [z, z+3e_2] \subset \mathbb{Z}_{\Omega^+}\} \le C \max\{|\diamondsuit_z^{(0,3)} f| : [z, z+3e_2] \subset \mathbb{Z}_\Omega\}$$

as Theorem 6.7 would require.

Similarly, in Figure 6.18, no extension of f to $\mathbb{Z}_{\Omega^+}$ can satisfy

$$\max\{|\diamondsuit_z^{(3,0)} f^+| : [z, z+3e_1] \subset \mathbb{Z}_{\Omega^+}\} \le C \max\{|\diamondsuit_z^{(3,0)} f| : [z, z+3e_1] \subset \mathbb{Z}_\Omega\}$$

as required in Theorem 6.7. □

Perhaps Theorem 6.7 fails in Figure 6.18 because to expand Ω would require that we append a group of cells bordering Ω in three directions—left, right, and either up or down—so that the new data point will be involved in divided differences with the existing data to its left, to its right, and either above or below. It may be that those are just too many new divided differences to keep bounded. In contrast, when we append a cell to the Ω in Figure 6.13, or even in Figure 6.16, the new data point is differenced with old data only to its left and below in an apparently manageable number of divided differences.

To speculate further, Theorem 6.7 fails in Figure 6.17 for the different reason that Ω violates Condition 4.15 in two directions—horizontal and vertical—in a very small area. The larger set in Figure 4.9 violates Condition 4.15 in two directions but is still salvageable by Theorem 6.1 because we can satisfy 4.15 by adding cells to parts of Ω that are some distance apart (and border Ω in only two directions).

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