Favard Interpolation from Subsets of a Rectangular Lattice

June 24, 2011

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Abstract

This is a study of Favard interpolation—in which the nth derivatives of the interpolant are bounded above by a constant times the nth divided differences of the data—in the case the data is given on some subset of a rectangular lattice in $\mathbb{R}^k$. In some instances, depending on the geometry of this subset, we construct a Favard interpolant, and in other instances, we prove that none exists.

Key Words and Phrases: interpolation, divided differences, multivariate, Favard.

1. Introduction

If f is a real-valued function defined either on an infinite set of real numbers

$$\mathbf{m} = \{\cdots < m_{-1} < m_0 < m_1 < \cdots\}$$

or on a finite set

$$\mathbf{m} = \{m_0 < m_1 < \cdots < m_d\},$$

then Rolle's Theorem implies that any smooth extension Ff of f must satisfy

$$\|D^n Ff\|_{L_\infty} \geq n!\|[m_i, \ldots, m_{i+n}]f\|_{\ell_\infty}.$$

Here, $[m_i, \ldots, m_{i+n}]f$ is the nth order divided difference of f formed from its values at $m_i, \ldots, m_{i+n}$, the L_∞ norm is over the interval $(\inf \mathbf{m}, \sup \mathbf{m})$, and the ℓ_∞ norm is taken over the collection of all allowable i (depending on the cardinality of $\mathbf{m}$). If the right side is finite, is there an extension Ff satisfying

$$\|D^n Ff\|_{L_\infty} \leq C(n)\|[m_i, \ldots, m_{i+n}]f\|_{\ell_\infty} \tag{1.1}$$

for some quantity $C(n)$ depending only on n?

Favard [1] answered this question in the affirmative when he showed that, for every such $\mathbf{m}$, f, and natural number n, not only does such a smooth function Ff exist, but it depends linearly and locally on f in such a way that, even if the divided differences of f are unbounded, $|D^n Ff|$ is locally no larger than the numbers

$$|[m_i, \ldots, m_{i+n}]f|$$

times some $C(n)$.

(In fact, it has been proven [6] that Favard's interpolant satisfies both (1.1) and

$$\|D^{n-1}Ff\|_{L_\infty} \le C(n)\|[m_i, \ldots, m_{i+n-1}]f\|_{\ell_\infty}$$

simultaneously, and that no interpolant can bound more derivatives by the corresponding divided differences on general $\mathbf{m}$.)

This is the author's fourth paper resulting from investigations into Favard's theorem and analogous theorems for functions f given on a discrete set of points in $\mathbb{R}^k$. In any such theorem, some derivative or collection of derivatives would have to take the place of the nth derivative of the function Ff and some differences would have to serve as the "multivariate" divided differences of order n of f. Until we establish more concrete notation later, we will refer loosely to the former as $(Ff)^{(n)}$ and to the latter as $f^{(n)}$. For instance, $(Ff)^{(n)}$ might consist of all mixed partial derivatives of total order n of Ff, and $f^{(n)}$ could be some derivative information gleaned from f by means of linear functionals that annihilate the k-variate polynomials of degree less than n. **Favard's interpolation problem** is to find an extension Ff of f that satisfies

$$|(Ff)^{(n)}| \le C(n,k)|f^{(n)}|. \tag{1.2}$$

This differs from the generalized Whitney Extension Theorem [8: VI.2, Theorem 4], which, among other things, promises that if f and its derivatives of total order $\le n$ are given on a closed set in $\mathbb{R}^k$, then there exists an extension $\mathcal{E}f$ of f to all $\mathbb{R}^k$ for which

$$\max_{h\le n}|(\mathcal{E}f)^{(h)}| \le C(n,k)\max_{h\le n}|f^{(h)}|. \tag{1.3}$$

To use the Whitney theorem in Favard's problem, given only the function f on a discrete set of data points, one would have to assign values to f's derivatives of order $\le n$ at the data points, presumably by means of finite differences. But even then, (1.3) will not guarantee that $\mathcal{E}f$ satisfies the more restrictive inequality (1.2).

Throughout this paper, $f^{(n)}$ will be taken to be the values at f of certain tensor product divided differences. This means that f must be defined on some subset of a **tensor product grid** of points in $\mathbb{R}^k$, that is, the Cartesian product of k sequences of real numbers

$$\mathbf{m}_i = \left(\cdots < m_{i,-1} < m_{i,0} < m_{i,1} < \cdots\right) \qquad (i = 1, \ldots, k).$$

In case each sequence $\mathbf{m}_i$ has constant step size Δ_i, in which case the Cartesian product is a rectangular **lattice**, a simple construction has been proven [4] to produce an interpolant Ff that depends linearly and locally on f and for which

$$|(Ff)^{(n)}| \le C(n,k)|f^{(n)}|$$

where $C(n,k)$ depends solely on n and k (and not on the step sizes $\Delta_1, \ldots, \Delta_k$ of the lattice). Later [6], it was discovered that no such results are possible on a tensor product grid without uniform step size in each direction; in other words, to bound derivatives by tensor product divided differences, we need the function f to be given on a lattice.

The focus of this paper is Favard's interpolation problem in case f is given on a subset (called the data set) of a rectangular lattice. In a way, it is the multivariate version of Favard's original problem in case the data set is finite, but what makes this problem different from that one and different from the multivariate problems studied earlier [4, 6] is the wide variety of possible data sets, the geometry of which will determine the existence of solutions to the interpolation problem.

In two papers [2, 3], Holtby used a multivariate Favard-like theorem from [4] to arrive at bounds on solutions to multivariate difference equations. The results presented here have potential for similar applications.

The basic problem and a necessary condition for it to have a solution are spelled out in Section 3. A new proof in Section 4 of an old result will allow us to prove in Section 5 that a simple geometric property is sufficient for the construction of a local solution to the interpolation problem.

We begin by establishing some notation in Section 2.

2. Notation

The ith component of a point x in $\mathbb{R}^k$ is denoted $x(i)$. If M is a $k \times k$ matrix, then the image of $x \in \mathbb{R}^k$ under M is the product Mx of M and (the column vector) x. If X is a subset of $\mathbb{R}^k$, then $MX := \{Mx : x \in X\}$.

For x and y in $\mathbb{R}^k$, let $[x, y]$ be the closed set of all u in $\mathbb{R}^k$ for which $x \le u \le y$ component-wise. The elements of $\mathbb{Z}^k$ are called **multi-integers**. In case x and y are in $\mathbb{Z}^k$, let $\{x, \ldots, y\}$ denote all the multi-integers in $[x, y]$. Let $\mathbb{1}$ denote the vector $(1, 1, \ldots, 1)$. If z is a multi-integer, then the set $[z, z + \mathbb{1}]$ is called a **cell**. For instance, a cell in $\mathbb{R}^3$ is a closed cube of volume one with multi-integer vertices.

If Ω is a set in $\mathbb{R}^k$, let $\mathbb{Z}_\Omega$ denote the set of multi-integers in Ω.

The coordinate direction vector e_i is the ith column of the $k \times k$ identity matrix.

The set of **multi-indices**, i.e., those multi-integers with nonnegative components, is written $\mathbb{Z}^k_+$. If α is a multi-index, then $\alpha! := \prod_i(\alpha(i)!)$ and $|\alpha| := \sum_i \alpha(i)$. Define the differential operator

$$D^\alpha := \left(\frac{\partial}{\partial x(1)}\right)^{\alpha(1)} \cdots \left(\frac{\partial}{\partial x(k)}\right)^{\alpha(k)}.$$

If f is a univariate function, then $[x_0, \ldots, x_n]f$ shall denote the nth divided difference of f at the real numbers $\{x_0, \ldots, x_n\}$, with the usual meaning if some $x_i = x_j$. When $z \in \mathbb{Z}^k$ and $\alpha \in \mathbb{Z}^k_+$, define $\Diamond^\alpha_z$ to be the **tensor product divided difference** that acts on k-variate functions by applying $[z(i), z(i)+1, \ldots, z(i)+\alpha(i)]$ in the ith variable for each $i = 1, \ldots, k$. For any positive diagonal matrix M, define $\Diamond^\alpha_{M,z} : f \mapsto \operatorname{diag}(M)^{-\alpha} \Diamond^\alpha_z\big(f(M\cdot)\big)$.

The **total order** of D^α and of $\Diamond^\alpha_z$ is $|\alpha|$. The polynomials of total degree less than n are those functions on $\mathbb{R}^k$ annihilated by every D^α of total order n.

3. The problem and a necessary condition for its solution

Problem 3.1. *Let n and k be positive integers, and let Ω be a connected union of cells in $\mathbb{R}^k$. Let M be a positive diagonal matrix. Find an operator $F_{\Omega,M}$ mapping functions f defined on $M\mathbb{Z}_\Omega$ to functions $F_{\Omega,M}f$ possessing all derivatives of total order n on $M\Omega$ so that*

$$F_{\Omega,M}f = f \text{ on } M\mathbb{Z}_\Omega \tag{3.2}$$

and

$$\max_{|\alpha|=n} \|D^\alpha F_{\Omega,M}f\|_{L_\infty(M\Omega)} \leq C \max\left\{\|\Diamond^\alpha_{M,z} f\| : |\alpha| = n,\ [z, z+\alpha] \subset \Omega\right\} \tag{3.3}$$

for some constant C independent of f.

Problem 3.1 might have no solution, as we will see in Lemma 3.5.

In the special case that $M = I$, we will write $F_{\Omega,M}$ as F_Ω.

The connectivity in Problem 3.1 is the same connectivity that one encounters in elementary topology [7; §23]. Note, however, that a connected union of cells must also be path-connected [7; §24].

The restriction $[z, z+\alpha] \subset \Omega$ imposed on the differences on the right side of (3.3) is motivated by the fact that, if $F_{\Omega,M}f$ is sufficiently smooth, then $M[z, z+\alpha]$ contains a point μ for which $(1/\alpha!)D^\alpha F_{\Omega,M}f(\mu) = \Diamond^\alpha_{M,z}F_{\Omega,M}f = \Diamond^\alpha_{M,z}f$. That is, the differences $\Diamond^\alpha_{M,z}f$ for which $[z, z+\alpha] \subset \Omega$ give a necessary minimum size to the αth derivative of any extension of f to $M\Omega$.

Definition 3.4. *Let n be a positive integer and let λ be a linear functional of the form*

$$\lambda : f \mapsto \sum_{s \in S} \lambda(s) f(s)$$

for some finite subset S of $\mathbb{R}^k$ called the **support** *of λ and associated scalars $\{\lambda(s) : s \in S\}$. We say that λ is an n***th difference** *if $\lambda p = 0$ for all polynomials of total degree less than n.*

For example, those tensor product divided differences of total order $\geq n$ that consist solely of function evaluations are nth differences.

Lemma 3.5. *Let Ω be a connected union of cells in $\mathbb{R}^k$ and let M be a positive diagonal matrix. If there exists an nth difference supported on $\mathbb{Z}_\Omega$ that is not in the span of*

$$\{\Diamond^\alpha_z : |\alpha| = n, [z, z+\alpha] \subset \Omega\}, \tag{3.6}$$

then no operator $F_{\Omega,M}$ can satisfy both (3.2) and (3.3).

Proof: Suppose there exists an nth difference λ supported on $\mathbb{Z}_\Omega$ that is not a linear combination of the differences in (3.6). Then the functional

$$\lambda_M : f \mapsto \lambda(f \circ M^{-1})$$

is an nth difference supported on $M\mathbb{Z}_\Omega$, and λ_M is not a linear combination of

$$\left\{\diamondsuit^\alpha_{M,z} : |\alpha| = n, [z, z+\alpha] \subset \Omega\right\}. \tag{3.7}$$

Consequently, there exists a function f defined on $M\mathbb{Z}_\Omega$ such that $\lambda_M f \neq 0$ but $\diamondsuit^\alpha_{M,z} f = 0$ for every $\diamondsuit^\alpha_{M,z}$ in (3.7).

Suppose that $F_{\Omega,M} f$ possesses all derivatives of order n everywhere on $M\Omega$, and that (3.2) and (3.3) are both true. Then the nth order derivatives of $F_{\Omega,M} f$ must be identically zero on $M\Omega$. Taylor's Theorem (below) and the path-connectedness of $M\Omega$ together imply that $F_{\Omega,M} f$ is a polynomial of total degree less than n, so that $\lambda_M F_{\Omega,M} f = 0$. On the other hand, (3.2) implies $\lambda_M F_{\Omega,M} f = \lambda_M f \neq 0$, a contradiction. ■

Taylor's Theorem 3.8. *If $g : \mathbb{R}^k \to \mathbb{R}$ possesses all derivatives of order n along a (possibly nonlinear) path in $\mathbb{R}^k$ from a to b, and if T_{n-1} is the Taylor polynomial of degree $n-1$ of g centered at a, then*

$$g(b) = T_{n-1}(b) - \int_a^b \sum_{|\alpha|=n} \frac{1}{\alpha!} D^\alpha g(x)\, d(b-x)^\alpha \tag{3.9}$$

where $\int_a^b$ is the integral along the hypothesized path and x is the variable of integration.

The inductive proof of Theorem 3.8 is left to the reader. To illustrate the notation in (3.9), if $k = 2$ and $\alpha = (2, 1)$, then

$$d(b-x)^\alpha = -2\big(b(1) - x(1)\big)\big(b(2) - x(2)\big)dx(1) - \big(b(1) - x(1)\big)^2 dx(2).$$

As one would expect, when $n = 1$, (3.9) is simply the fundamental theorem of calculus for path integrals:

$$g(b) = g(a) - \int_a^b \sum_{i=1}^k D^{e_i} g(x)\, d\big(b(i) - x(i)\big) = g(a) + \int_a^b \operatorname{grad} g(x) \cdot dx.$$

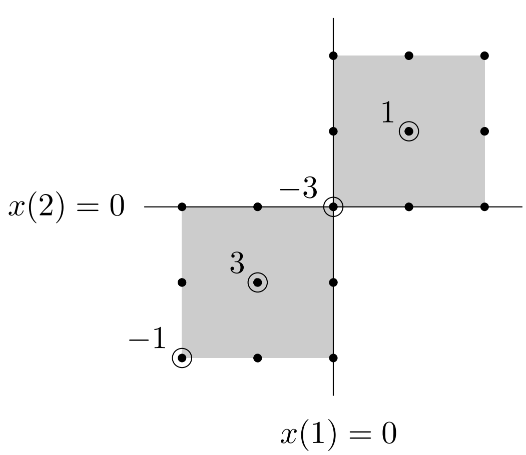

Figure 3.10 ($n = 3$)

Example 3.11: The set Ω shaded in Fig. 3.10 satisfies the hypotheses of Lemma 3.5 with $n = 3$. The integer points $\mathbb{Z}_\Omega$ of Ω are marked $\bullet$, and four of these (marked $\bigcirc$) support a third difference λ; the numbers $\lambda(s)$ are marked beside the support points s. Along the support of any one of the differences in (3.6), the function

$$f(x) := \begin{cases} 0 & \text{if } x(2) \le 0 \\ x(1)x(2) & \text{if } x(2) > 0 \end{cases}$$

is a polynomial of degree less than 3. Consequently, each of the differences in (3.6) annihilates f. On the other hand, $\lambda f = 1$. Therefore λ is not in the span of (3.6), and Problem 3.1 has no solution F_Ω on Ω. □

4. The tensor product construction

An earlier theorem [4, Theorem 4.3] solves Favard's interpolation problem on $\mathbb{Z}^k$. In this section, we present a more general construction solving the same problem and having applications to the main result in Section 5.

Fix positive integers n and k and a univariate function ψ satisfying

$$\text{(4.1)} \qquad \psi \in C^\infty(\mathbb{R}), \qquad \operatorname{supp}\psi \subseteq [-1,1], \text{ and } \qquad \sum_{z\in\mathbb{Z}} \psi(\cdot - z) = 1.$$

For every integer j, let P_j be the interpolation operator that maps each univariate function g to the polynomial $P_j g$ of degree less than n agreeing with g at the points $\{j, \ldots, j+n-1\}$.

For each i in $\{1, \ldots, k\}$ and z in $\mathbb{Z}$, choose an integer $j_{i,z}$ such that

$$j_{i,z} \le z < j_{i,z} + n.$$

(In the proof of [4, Theorem 4.3], each $j_{i,z}$ was taken to be z.)

Define the operators

$$E_i := \sum_{z\in\mathbb{Z}} \psi(\cdot - z) P_{j_{i,z}},$$

which maps functions defined on $\mathbb{Z}$ to functions defined on $\mathbb{R}$, and

$$H := E_1 \otimes E_2 \otimes \cdots \otimes E_k,$$

which maps functions defined on $\mathbb{Z}^k$ to functions defined on $\mathbb{R}^k$.

Theorem 4.2. *If n, k, ψ, and H are as above, and if f is any function defined on $\mathbb{Z}^k$, then the function Hf has the following properties.*

(4.3) $Hf \in C^\infty(\mathbb{R}^k)$.
(4.4) $Hf = f$ on $\mathbb{Z}^k$.
(4.5) Hf *depends linearly on* f.

(4.6) *Hf depends locally on f; specifically, the restriction of Hf to any cell $[u, u+\mathbb{1}]$ depends entirely on the values of f at the multi-integers $\{u-(n-1)\mathbb{1}, \ldots, u+n\mathbb{1}\}$.*

(4.7) *There exists a constant $C(n,k,\psi)$ such that for all functions $f:\mathbb{Z}^k \to \mathbb{R}$ and all multi-integers u and all multi-indices $\alpha \le n\mathbb{1}$,*

$$\begin{aligned}&\max\left\{|D^\alpha Hf(x)| : x \in [u, u+\mathbb{1}]\right\}\\ &\quad \le C(n,k,\psi)\max\left\{|\Diamond_z^\alpha f| : u-(n-1)\mathbb{1} \le z \le u+n\mathbb{1}-\alpha\right\}.\end{aligned}$$

Proof of Theorem 4.2:

Note that

$$H = \sum_{z\in\mathbb{Z}^k} \Psi(\cdot - z) Q_z, \tag{4.8}$$

where Ψ is the tensor product of k copies of ψ, and Q_z is the interpolation operator mapping any function f to the polynomial $Q_z f$ of degree less than n in each variable that agrees with f at the n^k multi-integers q for which

$$j_{i,z(i)} \le q(i) < j_{i,z(i)} + n$$

for every $i \in \{1, \ldots, k\}$.

Properties (4.3)–(4.6) are immediate. To prove (4.7), renumber, if necessary, to make $u = 0$.

Observe that

$$E_i = \psi P_{j_{i,0}} + \psi(\cdot - 1) P_{j_{i,1}}$$

on $[0,1]$, in the sense that the interpolation operators on both sides produce the same function on the interval $[0,1]$. Letting $j_{i\downarrow}$ and $j_{i\uparrow}$ denote the minimum and maximum, respectively, of $j_{i,0}$ and $j_{i,1}$,

$$\begin{aligned}E_i &= P_{j_{i,0}} + \psi(\cdot - 1)(P_{j_{i,1}} - P_{j_{i,0}}),\\ &= P_{j_{i,0}} \pm \psi(\cdot - 1)\sum_{l=j_{i\downarrow}}^{j_{i\uparrow}-1}(P_{l+1} - P_l).\end{aligned}$$

(The above sum is empty if $j_{i\downarrow} = j_{i\uparrow}$.) Expanding the operators P in terms of Newton polynomials and divided differences gives

$$\begin{aligned}E_i =& \sum_{\ell=0}^{n-1}\big(\cdot - j_{i,0}\big)\cdots\big(\cdot - (j_{i,0}+\ell-1)\big)[j_{i,0}, j_{i,0}+1, \ldots, j_{i,0}+\ell]\\ &\pm \psi(\cdot - 1)\sum_{l=j_{i\downarrow}}^{j_{i\uparrow}-1}\big(\cdot - (l+1)\big)\cdots\big(\cdot - (l+n-1)\big)n[l, \ldots, l+n].\end{aligned}$$

Consequently, H can be written on $[0, \mathbb{1}]$ as the sum

$$H = \sum_{\substack{\beta \le n\mathbb{1} \\ -n\mathbb{1} < \zeta \le 0}} \Phi_{\zeta,\beta} \diamondsuit_\zeta^\beta$$

in the sense that the interpolation operators on both sides produce the same function on $[0, \mathbb{1}]$, where each $\Phi_{\zeta,\beta}$ either is a tensor product of polynomials and polynomials times shifts of ψ or is identically zero.

The rest of the proof is the same as its original form [4, Theorem 4.3]:

Fix f and α as in the hypothesis. If $\beta(i) < \alpha(i)$ for some i, then $\beta(i) < n$ and, in the ith variable, $\Phi_{\zeta,\beta}$ is a polynomial of degree $\beta(i)$. Hence,

$$D^\alpha H f = \sum_{\substack{\alpha \le \beta \le n\mathbb{1} \\ -n\mathbb{1} < \zeta \le 0}} D^\alpha \Phi_{\zeta,\beta} \diamondsuit_\zeta^\beta f.$$

Express each $\diamondsuit_\zeta^\beta$ as a linear combination of those $\diamondsuit_z^\alpha$ for which $[z, z+\alpha] \subset [\zeta, \zeta+\beta]$. Note that there are only finitely many different $\Phi_{\zeta,\beta}$ possible (depending on the choice of $j_{i,0}$ and $j_{i,1}$), and (4.7) follows. ■

5. A sufficient condition for a general solution

In this section, we see that the Problem 3.1 has a local and linear solution when Ω satisfies the following geometric condition.

Condition 5.1. *For every integer i in $\{1, \ldots, k\}$ and every integer c and every connected component E of $\{x \in \Omega : x(i) = c\}$, there exists an integer j_E in $\{c-n+1, \ldots, c\}$ so that, for every point x in E,*

$$x + (j_E - x(i))e_i + e_i[0, n-1] \subset \Omega. \tag{5.2}$$

(That is, the closed line segment from the point

$$\big(x(1), \ldots, \overset{j_E}{\cancel{x(i)}}, \ldots, x(k)\big)$$

to the point

$$\big(x(1), \ldots, \overset{j_E+n-1}{\cancel{x(i)}}, \ldots, x(k)\big)$$

is contained entirely in Ω.)

Example 5.3: Suppose that Ω is the closed shaded region in Fig. 5.4 and that $\bullet$s mark the integer points $\mathbb{Z}_\Omega$ of Ω. Then Ω satisfies Condition 5.1 in case $n = 3$. For instance, let E be the set of points x in Ω at which $x(2) = 6$. Condition 5.1 requires j_E to be one of $\{4, 5, 6\}$, and if we take $j_E = 6$, then the closed vertical line segment from the point $\big(x(1), j_E\big)$ to the point $\big(x(1), j_E + 2\big)$ lies entirely in Ω for every x in E. Fig. 5.4 includes some of these vertical line segments.

Note that Condition 5.1 allows us to choose a different integer in $\{1, 2, 3\}$ for each of the connected components E' and E'' of $\{x \in \Omega : x(2) = 3\}$; in fact, the only possible choices are $j_{E'} = 1$ and $j_{E''} = 3$. Without going into further detail, Ω satisfies Condition 5.1 when $n = 3$ because every horizontal (vertical) line segment in Ω passing through a point in $\mathbb{Z}_\Omega$ is contained in a rectangle of vertical (horizontal) diameter 2.

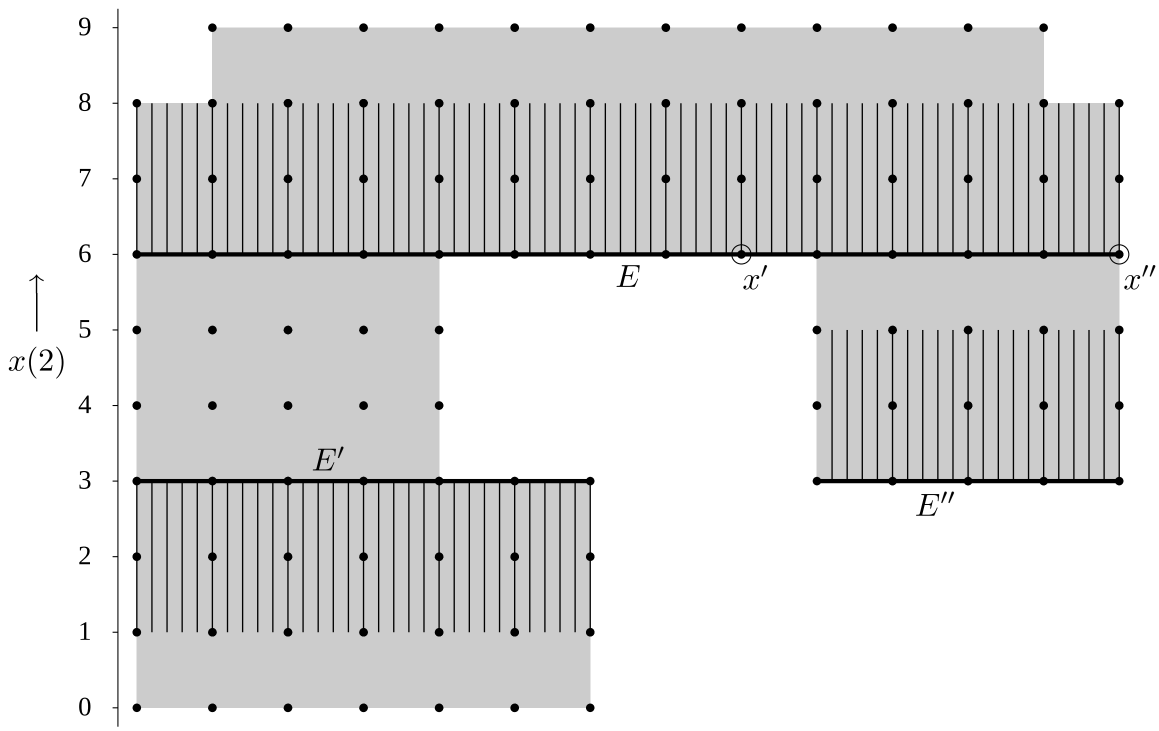

Figure 5.4 ($n = 3$, but not 4)

On the other hand, Ω violates Condition 5.1 when $n = 4$ because there is no integer j_E in $\{3, 4, 5, 6\}$ for which the closed vertical line segment from the point $\big(x(1), j_E\big)$ to the point $\big(x(1), j_E + 3\big)$ lies entirely in Ω for every x in E. For instance, $x = x'$ requires $j_E = 6$, but $x = x''$ requires $j_E \leq 5$.

□

Theorem 5.5. *Let ψ be a function satisfying (4.1). If Ω is a connected union of cells that satisfies Condition 5.1, then, for every positive diagonal $k \times k$ matrix M, there exists an operator $F_{\Omega,M}$ mapping functions f defined on $M\mathbb{Z}_\Omega$ to functions $F_{\Omega,M}f$ with the following properties.*

(5.6) $F_{\Omega,M} f \in C^\infty(M\Omega)$.
(5.7) $F_{\Omega,M} f = f$ on $M\mathbb{Z}_\Omega$.
(5.8) *$F_{\Omega,M} f$ depends linearly on f.*
(5.9) *$F_{\Omega,M} f$ depends locally on f; specifically, for any cell $[u, u + \mathbb{1}] \subset \Omega$, the restriction of $F_{\Omega,M} f$ to $M[u, u + \mathbb{1}]$ depends entirely on the values of f at the points of $M\mathbb{Z}_\Omega$ that lie in the same connected component of $M\big([u - (n-1)\mathbb{1}, u + n\mathbb{1}] \cap \Omega\big)$ as Mu.*
(5.10) *There exists a constant $C(n, k, \psi)$ such that, for any multi-index $\alpha \le n\mathbb{1}$ and any cell $[u, u + \mathbb{1}] \subset \Omega$,*

$$\max \{|D^\alpha F_{\Omega,M} f(x)| : x \in M[u, u + \mathbb{1}]\} \le C(n, k, \psi) \max{}^* |\diamondsuit^\alpha_{M,v} f|,$$

where $\max^*$ is taken over those $v \in \mathbb{Z}_\Omega$ for which $[v, v + \alpha]$ lies in the same connected component of $[u - (n-1)\mathbb{1}, u + n\mathbb{1}] \cap \Omega$ as u.

Note that the constant $C(n, k, \psi)$ is independent of the matrix M.

Example 5.11: Let $M = I$ and Ω be the shaded figure in Fig. 5.12, whose integer points $\mathbb{Z}_\Omega$ are marked with •s, and assume $n = 3$. It is straightforward to verify that Ω satisfies Condition 5.1.

Let $[u, u+\mathbb{1}]$ be the darker square. The set $[u-2\mathbb{1}, u+3\mathbb{1}]$ is outlined with the dashed line. The integer points in the same connected component of $[u - 2\mathbb{1}, u + 3\mathbb{1}] \cap \Omega$ as u are marked ◯. The points marked □ are also in $[u - 2\mathbb{1}, u + 3\mathbb{1}] \cap \Omega$ but are not in the same connected component as u. According to (5.9), the restriction of the interpolant $F_\Omega f$ to $[u, u + \mathbb{1}]$ depends only on the values of f at the ◯ points. The differences appearing on the right side of (5.10) are those αth differences supported on the ◯ points. □

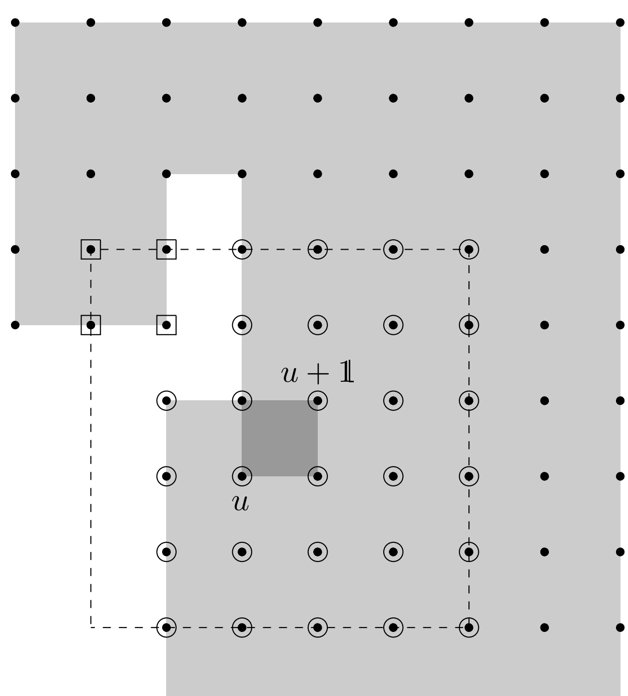

Figure 5.12 ($n = 3$)

Proof of Theorem 5.5: It is necessary only to construct $F_{\Omega,M}$ in case M is the $k \times k$ identity matrix (in which case $F_{\Omega,M}$ is written F_Ω), since then the operator $F_{\Omega,M}$ obtained from F_Ω by scaling,

$$F_{\Omega,M} f := \big(F_\Omega(f \circ M)\big) \circ M^{-1},$$

inherits properties (5.6)–(5.10).

Let Ψ be the tensor product of k copies of the univariate function ψ. We'll construct F_Ω to be an operator of the form

$$F_\Omega = \sum_{z \in \mathbb{Z}_\Omega} \Psi(\cdot - z) R_z$$

where each R_z is a (tensor product) linear interpolation operator mapping f to a k-variate polynomial $R_z f$ of degree less than n in each variable that agrees with f at z and $n^k - 1$ other points (yet to be specified) in $\{z - (n-1)\mathbb{1}, \ldots, z + (n-1)\mathbb{1}\}$. Properties (5.6)–(5.8) will then follow.

For each $i \in \{1, \ldots, k\}$ and $c \in \mathbb{Z}$ and E a connected component of $\{x \in \Omega : x(i) = c\}$, choose an integer j_E as in Condition 5.1. For each $z \in \Omega$, let $E(z,i)$ be the connected component of $\{x \in \Omega : x(i) = z(i)\}$ containing z. Take the n^k interpolation points of R_z to be those multi-integers q satisfying

$$j_{E(z,i)} \le q(i) < j_{E(z,i)} + n$$

for every $i \in \{1, \ldots, k\}$. All such q necessarily lie in $\{z - (n-1)\mathbb{1}, \ldots, z + (n-1)\mathbb{1}\}$. For $F_\Omega f$ to be well defined, it must be shown that these multi-integers q lie in Ω. We'll prove the stronger statement that if the multi-integer z lies in the cell $[u, u + \mathbb{1}] \subset \Omega$, then the convex hull of the interpolation points for R_z also lies in Ω. This will prove both the well-definedness of $F_\Omega f$ and (5.9).

For each $i \in \{1, \ldots, k\}$, the cell $[u, u + \mathbb{1}]$ has only two faces perpendicular to e_i, one passing through u and the other through $u + \mathbb{1}$. The first of these belongs to $E(u,i)$ and the second to $E(u + \mathbb{1}, i)$. Therefore, if $z \in \{u, \ldots, u + \mathbb{1}\}$, then the ith components of the interpolation points for R_z must be either

$$j_{E(u,i)}, \ldots, j_{E(u,i)} + n - 1$$

or

$$j_{E(u+\mathbb{1},i)}, \ldots, j_{E(u+\mathbb{1},i)} + n - 1.$$

Let $m_\downarrow(i) := \min\big(j_{E(u,i)}, j_{E(u+\mathbb{1},i)}\big)$, and let $m_\uparrow(i) := \max\big(j_{E(u,i)}, j_{E(u+\mathbb{1},i)}\big) + n - 1$. We will show that

$$[m_\downarrow(1), m_\uparrow(1)] \times [m_\downarrow(2), m_\uparrow(2)] \times \cdots \times [m_\downarrow(k), m_\uparrow(k)] \subset \Omega$$

by proving that, for all $r \in \{1, \dots, k\}$,

$$\begin{aligned}(5.13)\qquad &[m_\downarrow(1), m_\uparrow(1)] \times \cdots \times [m_\downarrow(r), m_\uparrow(r)] \\ &\qquad \times [u(r+1), u(r+1)+1] \times \cdots \times [u(k), u(k)+1] \subset \Omega.\end{aligned}$$

Induct on r, the case $r = 0$ being trivial. Assume (5.13) is true for some $r < k$. Since the face

$$\begin{aligned}&[m_\downarrow(1), m_\uparrow(1)] \times \cdots \times [m_\downarrow(r), m_\uparrow(r)] \\ &\qquad \times \{u(r+1)\} \\ &\qquad \times [u(r+2), u(r+2)+1] \times \cdots \times [u(k), u(k)+1]\end{aligned}$$

lies entirely in $E(u, r+1)$, the choice of $j_{E(u,r+1)}$ and (5.2) ensures that

$$\begin{aligned}(5.14)\qquad &[m_\downarrow(1), m_\uparrow(1)] \times \cdots \times [m_\downarrow(r), m_\uparrow(r)] \\ &\qquad \times [j_{E(u,r+1)}, j_{E(u,r+1)} + n - 1] \\ &\qquad \times [u(r+2), u(r+2)+1] \times \cdots \times [u(k), u(k)+1] \subset \Omega.\end{aligned}$$

Similarly,

$$\begin{aligned}(5.15)\qquad &[m_\downarrow(1), m_\uparrow(1)] \times \cdots \times [m_\downarrow(r), m_\uparrow(r)] \\ &\qquad \times [j_{E(u+\mathbb{1},r+1)}, j_{E(u+\mathbb{1},r+1)} + n - 1] \\ &\qquad \times [u(r+2), u(r+2)+1] \times \cdots \times [u(k), u(k)+1] \subset \Omega.\end{aligned}$$

Recalling that

$$j_{E(u,r+1)} \le u(r+1) \le j_{E(u,r+1)} + n - 1$$

and

$$j_{E(u+\mathbb{1},r+1)} \le u(r+1) + 1 \le j_{E(u+\mathbb{1},r+1)} + n - 1,$$

taking the union of the left sides of (5.13), (5.14), and (5.15) gives (5.13) with r replaced by $r+1$, completing the inductive step and the proof of both (5.9) and the well-definedness of $F_\Omega f$.

To prove (5.10), observe that, globally, F_Ω may differ from the tensor product operator H constructed in the proof of Theorem 4.2, since, in H, the ith components of the interpolation points

$$j_{i,z(i)}, \dots, j_{i,z(i)} + n - 1$$

of Q_z depend only on $z(i)$. (In contrast, if u and v belong to Ω and $u(i) = v(i)$, and if $E(u, i) \ne E(v, i)$, then it is possible that the ith components of the interpolation points for R_u differ from those of R_v.) Locally, however, the two operators are the same. Specifically, if v and w are both in $\{u, \dots, u+\mathbb{1}\}$ and satisfy $v(i) = w(i)$, then v and w necessarily belong to the same connected component of $\{x \in \Omega : x(i) = v(i)\}$, and so the ith components of the interpolation points of R_v and R_w are the same. Therefore, there exists an operator of the form (4.8) with $Q_z = R_z$ for all z in $\{u, \dots, u + \mathbb{1}\}$, and consequently $F_\Omega f = Hf$ on $[u, u + \mathbb{1}]$. (5.10) now follows from (4.7). ∎

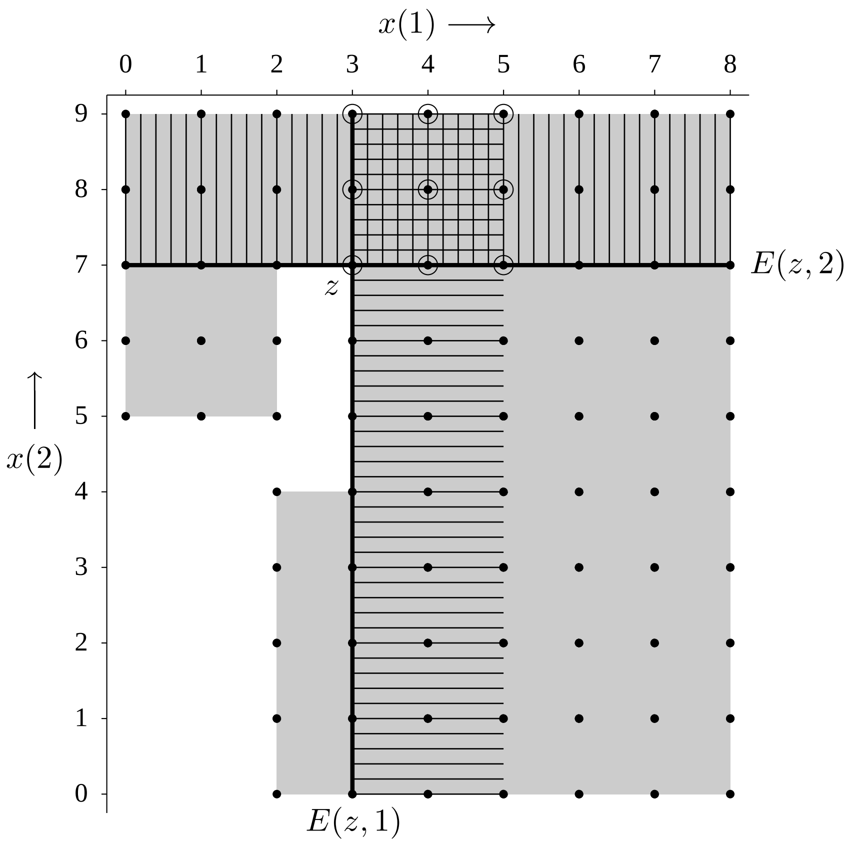

Figure 5.16 ($n = 3$)

Example 5.17: Let $n = 3$, and take Ω to be the shaded set in Fig. 5.16. We have labeled an integer z in $\mathbb{Z}_\Omega$ and the corresponding line segments $E(z,1)$ and $E(z,2)$. Note that the only integer $j_{E(z,1)}$ in $\{1,2,3\}$ for which the horizontal line segment from the point $\big(j_{E(z,1)}, x(2)\big)$ to the point $\big(j_{E(z,1)}+2, x(2)\big)$ lies in Ω for each x in $E(z,1)$ is $j_{E(z,1)} = 3$. We have drawn some of these horizontal line segments in Fig. 5.16. Similarly, in order for the vertical line segment from $\big(x(1), j_{E(z,2)}\big)$ to $\big(x(1), j_{E(z,2)}+2\big)$ to lie in Ω for all x in $E(z,2)$ and for some $j_{E(z,2)}$ in $\{5,6,7\}$, one must choose $j_{E(z,2)}$ to be 7. We have drawn some of these vertical line segments in the figure. The interpolation points of R_z are the multi-integers (marked with a $\bigcirc$) that lie in both the $\equiv$-shaded and the $||||$-shaded rectangles.

To illustrate further, in Fig. 5.18, we have labelled three other points u, v, and w from $\mathbb{Z}_\Omega$, and the corresponding line segments $E(u,1)$, $E(u,2)$, $E(v,1)$, ... One could take $j_{E(v,2)}$ to be either 3, 4, or 5, but choose it to be 4. The other j_E values in the table below are the only ones possible.

E	$E(u,1)$	$E(v,1) = E(w,1)$	$E(u,2)$	$E(v,2)$	$E(w,2)$
j_E	0	3	7	4	2

These j_E values determine the interpolation points of R_u (marked with a $\bigcirc$), of R_v ($\square$), and of R_w ($\triangle$). $\square$

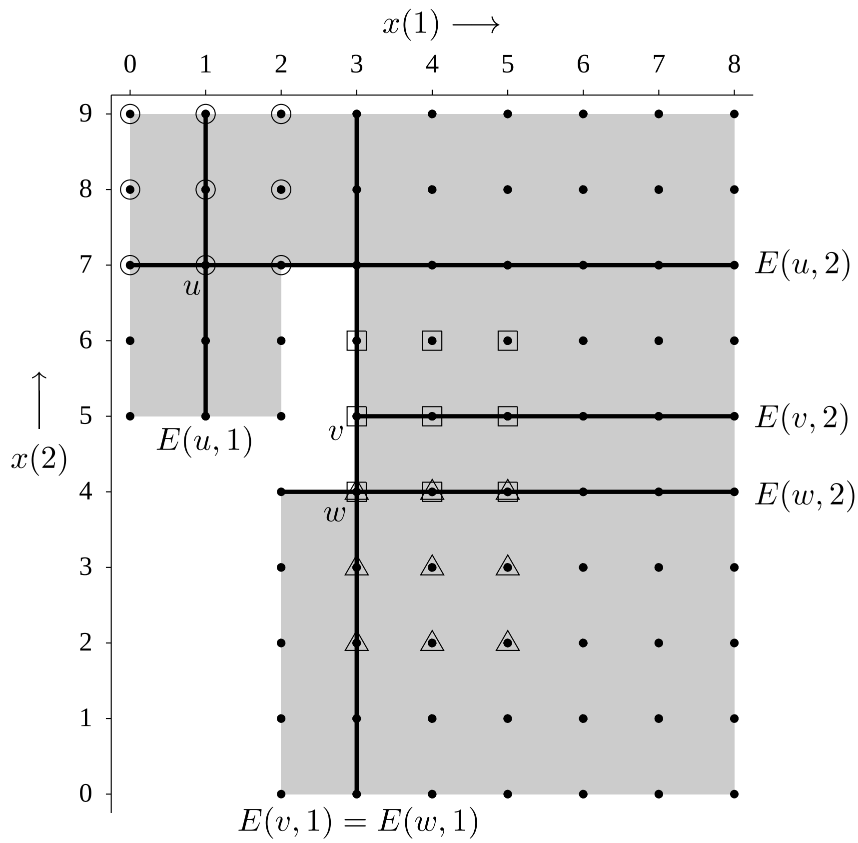

Figure 5.18 ($n = 3$**)**

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