

Flat extension and ideal projection

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Abstract

A generalization of the the flat extension theorems of Curto and Fialkow and Laurent and Mourrain is obtained by seeing the problem as one of ideal projection. Some other results on ideal projection are generalized, and it is seen that a linear functional on the k -variate polynomials over \mathbb{C} is finitely atomic if and only if its moment matrix has finite rank, a fact previously known for positive functionals. A characterization of solutions to constant-coefficient difference equations obtained as a corollary.

Key words: flat extension, ideal, projection, difference equations, multivariate, moment problem, moment matrix, Hankel operator.

1. Introduction

The Hamburger moment problem is to determine the existence and uniqueness of a positive measure whose moments

$$\int_{\mathbb{R}^k} ()^\alpha d\mu, \quad \alpha \in \mathbb{Z}_+^k.$$

take a prescribed sequence of values $\{y_\alpha\}_{\alpha \in \mathbb{Z}_+^k}$. Here, $()^\alpha$ is the α th monomial in Π , the vector space of all polynomials on \mathbb{R}^k . For such a measure to exist, it is necessary and sufficient that the linear functional

$$L : \Pi \rightarrow \mathbb{R} : p = \sum_{\alpha} \hat{p}(\alpha) ()^\alpha \mapsto \sum_{\alpha} \hat{p}(\alpha) y_\alpha$$

be nonnegative, that is, $Lp \geq 0$ if $p \geq 0$, and for this to occur, it is necessary that the **moment matrix**

$$[y_{\alpha+\beta}]_{\alpha, \beta \in \mathbb{Z}_+^k} \tag{1}$$

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be positive semidefinite, that is, $Lp^2 \geq 0$ for all p . This condition is sufficient only under special circumstances, for instance, when every positive polynomial can be written as a sum of squares, as occurs when $k = 1$ (Pólya and Szegő, 1976, §6.6) and again when (1) has finite rank, since in that case the functional L is finitely atomic (Curto and Fialkow, 1996, Theorem 7.7).

Finitely atomic measures and their moment matrices are the subject of the **truncated moment problem** of Curto and Fialkow (1996), who address when a such a measure is determined by finitely many of its moments. To be precise, Curto and Fialkow work in \mathbb{C}^k and study the relationship between the complex moment matrix

$$\left[\gamma_{\alpha, \beta} := \int_{\mathbb{C}^k} \bar{()^\alpha} ()^\beta d\mu \right]_{\alpha, \beta \in \mathbb{Z}_+^k}, \quad (2)$$

its submatrices of the form

$$[\gamma_{\alpha, \beta}]_{|\alpha|, |\beta| \leq n}, \quad (3)$$

and the size of the support of μ . One of their main results is that if the positive semidefinite (3) can be extended **flatly**, i.e., without increasing rank, to a positive semidefinite

$$[\gamma_{\alpha, \beta}]_{|\alpha|, |\beta| \leq n+1},$$

then the later has a unique, positive semidefinite, flat extension (2) and a finitely-atomic representing measure μ , and the rank of the matrix equals the cardinality of the measure's support.

Curto and Fialkow's flat extension theorem in the special case when \mathbb{C} is replaced by \mathbb{R} is generalized by Laurent and Mourrain (2009). Working with moment matrices of the form (1), they replace \mathbb{C} by any field \mathbb{F} and do not require moment matrices to be positive semidefinite. They then prove that if the finite set $C \subset \mathbb{Z}_+^k$ is **connected to one**, meaning

$$C \neq \emptyset, \text{ and } \alpha \in C \setminus 0 \implies \alpha - e_j \in C \text{ for some } j, \quad (4)$$

and if

$$[y_{\alpha+\beta}]_{\alpha, \beta \in C} \quad (5)$$

can be flatly extended to

$$[y_{\alpha+\beta}]_{\alpha, \beta \in C^+} \quad (6)$$

where

$$C^+ := \{\alpha + e_j : \alpha \in C, j = 0 : k\}$$

and $e_j \in \mathbb{F}^k$ is the vector given by

$$e_j(i) = \delta_{i,j}, \quad i = 1 : k, j = 0 : k, \quad (7)$$

then (6) has a unique flat extension (1). In addition, in case \mathbb{F} is \mathbb{R} , if (6) is positive semidefinite, then so is (1).

(To be consistent with Laurent and Mourrain, we'll say that the **monomial space** $\text{span}\{()^\alpha : \alpha \in C\}$ is connected to one if the set $C \subset \mathbb{Z}_+^k$ is connected to one.)

Extending a truncated moment sequence y from some finite $C \subset \mathbb{Z}_+^k$ to all of \mathbb{Z}_+^k simply means extending the domain of the linear functional L from a finite dimensional polynomial space to Π . Such extensions are always possible using functionals—though not necessarily positive measures—supported at finitely many points. But, in order that the moment matrix (1) be a flat extension of (5), the dependence relations among its columns force L to vanish on a polynomial ideal of finite codimension and impose on

the moments a set of homogeneous difference equations which, depending on the initial conditions (5), may or may not have a solution (Example 4.8). Consequently, (1) has finite rank exactly when $L = LP$ for some finite-rank **ideal projector** P , that is, a projector whose kernel is an ideal (Proposition 4.1), and questions of flat extension naturally draw on our understanding of such projections. In case $\mathbb{F} = \mathbb{C}$, (1) has finite rank if and only if L is finitely atomic, i.e., a linear combination of finitely many shifts of δ and its derivatives (Theorem 4.2), a fact which leads to a characterization of the solutions to multivariate homogeneous difference equations with constant coefficients (Corollary 3.4).

As shown by Boor (2005), an ideal projector onto the space B is completely determined by its restriction to

$$B^* := \Pi_0 + \Pi_1 \cdot B \tag{8}$$

(where Π_n is the space of polynomials of degree less or equal n). On the other hand, not every projector from B^* onto B can be extended to an ideal projection onto B , and Mourrain (1999) gives two conditions, one of which is B 's connection to one, that are sufficient for such an extension to exist. In Proposition 4.3, we see that when we replace connection to one with a new condition, the result is necessary and sufficient for the extension of a projection $B^* \rightarrow B$ to an ideal projection $\Pi \rightarrow B$. From there, it's an easy matter to generalize the flat extension theorem of Laurent and Mourrain, and Theorem 4.7 gives necessary and sufficient conditions for flat extension, doing away with the requirements that B be connected to one and have a monomial basis (which was never assumed in Mourrain's theorem.)

Shohat and Tamarkin (1943) is the standard reference for the classical moment problem. See Berg (1980) for a concise introduction to the multivariate moment problem and its solution via harmonic analysis.

When $k = 1$, a matrix of the form (1) is a Hankel matrix. See Iokhvidov (1982) for the general algebraic theory of flat extensions (a.k.a. singular extensions) of Hankel and Toeplitz matrices.

Writing Π as the direct sum of an ideal I and a finite dimensional vector space B is central to the subjects of ideal interpolation and the construction of ideal bases that facilitate the decomposition of a polynomial into its I - and B -components, i.e., its normal form. For example, see Boor (2005), Boor (2006) Boor and Ron (1991), Boor and Ron (1992), Gasca and Sauer (2000), Kehrein and Kreuzer (2005), Kehrein et al. (2005), Mourrain and Trebuchet (2005), Mourrain (1999), Möller and Sauer (2000a), Möller and Sauer (2000b), Sauer (1998), Sauer (2001), Shekhtman (2009).

Moment matrices for finitely atomic measures and the associated ideal bases have been applied to the optimization of polynomials and the solution of systems of polynomial equations. See Laurent and Rostalski (2012), Lasserre et al. (2009), Laurent (2005), Laurent (2007), for instance.

Analogous to the Hausdorff moment problem, the truncated K -moment [and flat extension] problem is to determine whether a truncated moment matrix has a representing [finitely atomic] measure supported on some compact semi-algebraic set K in \mathbb{C}^k (e.g., Curto and Fialkow, 2000) or in \mathbb{R}^k (e.g., Nie, 2014).

Moments of discrete positive measures and the interpolation of finitely many moments with discrete distributions are of interest both in probability and statistics (Adan et al., 1995), (Norton and Arnold, 1985) and in approximation theory, where the latter is known as cubature (Cools and Rabinowitz, 1993), (Cools, 1999).

2. Definitions and notation.

Here are most of the definitions, including some that are introduced elsewhere.

\mathbb{F} : a field.

k : a positive integer.

id_S : the identity operator on the set S ; “id” if the domain is clear from context.

1 : the constant polynomial 1.

\mathbb{Z}_+^k : the set of all k -tuples of nonnegative integers.

e_j : the vector in \mathbb{F}^k given by $e_j(i) := \delta_{i,j}$ for $i = 1 : k$ and $j = 0 : k$.

$\Pi(\mathbb{F}^k)$: the ring of polynomials in k variables over the field \mathbb{F} , or Π if the field is clear from context.

$()^\alpha$: the α th monomial for $\alpha \in \mathbb{Z}_+^k$.

monomial space: the span of a set of monomials.

$()_j := ()^{e_j}$, for $j = 0 : k$

multi-sequence: a function $\mathbb{Z}_+^k \rightarrow \mathbb{F}$.

E^α : the shift operator on multi-sequences given by $E^\alpha y := y(\cdot + \alpha)$.

For the polynomial $p =: \sum_\alpha \hat{p}(\alpha)()^\alpha$,

$\text{totdeg } p :=$ the **total degree** of the polynomial $p := \max\{|\alpha| : \hat{p}(\alpha) \neq 0\}$.

$p(D)$: the differential operator $\sum_\alpha \hat{p}(\alpha)D^\alpha$.

$p(E)$: the difference operator $\sum_\alpha \hat{p}(\alpha)E^\alpha$.

$\Pi_m :=$ the set of polynomials in Π having total degree less or equal n .

$\#U$: the cardinality of the set U .

$A \subseteq B :=$ A is a subspace of B , possibly equal to B .

$\text{var}(S)$: the variety generated by the set of functions S .

$\text{ide}(S)$: the ideal generated by $S :=$ either a set of polynomials or a variety.

$B \cdot C :=$ the vector space spanned by all products bc of elements $b \in B$ and $c \in C$.

$B^* := \Pi_0 + \Pi_1 \cdot B$

δ_x : the point-evaluation linear functional $p \mapsto p(x)$.

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

For $L \in \Pi'$:

moments of $L := \{L()^\alpha : \alpha \in \mathbb{Z}_+^k\}$.

moment matrix of L : (1).

H^L : the **Hankel operator** associated to L given by $H^L : b \mapsto L(b \cdot)$.

$I_L := \ker H^L = \{p \in \Pi : L(p\Pi) = 0\} = \{p \in \Pi : p(E)y = 0\}$, where $y_\alpha := L()^\alpha$.

ideal of **finite codimension**: one whose cosets form a finite dimensional vector space (a.k.a “zero-dimensional”).

ideal projector: a projector on Π whose kernel is an ideal.

${}^x \perp, \perp_x$, **multiplicity space**: see 3.2.

\xrightarrow{b} , **flat extension**: see (16).

$X, b(X)$: see (22), (23).

connected to one: see (4) and the paragraph which follows it.

3. Finitely supported functionals

In this section, we characterize the finitely supported functionals L on $\Pi(\mathbb{C}^k)$ as those for which $\text{var}(I_L)$ is finite. This will be part of a larger characterization theorem in Section 4.

Proposition 3.1. *Suppose U is a finite subset of \mathbb{C}^k , and for each x in U , q_x is a nonzero polynomial. If $L \in \Pi'$ has the form*

$$L : p \mapsto \sum_U \delta_x q_x(D)p, \quad (9)$$

then

$$I_L = \{p \in \Pi : \delta_x(D^\alpha q_x)(D)p = 0, x \in U, \alpha \in \mathbb{Z}_+^k\} \quad (10)$$

and $U = \text{var}(I_L)$. If, in addition, each q_x is constant, then I_L is radical.

Proof. The identity

$$\delta_x(r(D)q)(D)p = \delta_x q(D)(r(\cdot - x)p), \quad p, q \in \Pi, x \in \mathbb{C}^k \quad (11)$$

(Boor, 2005, p. 74) implies

$$\begin{aligned} & \forall x \in U, \alpha \in \mathbb{Z}_+^k, \quad \delta_x(D^\alpha q_x)(D)p = 0 \\ \Leftrightarrow & \forall x \in U, \alpha \in \mathbb{Z}_+^k, \quad \delta_x((D+x)^\alpha q_x)(D)p = 0 \\ \Leftrightarrow & \forall x \in U, \alpha \in \mathbb{Z}_+^k, \quad \delta_x(q_x)(D)((\cdot)^\alpha p) = 0 \\ & \Rightarrow \forall \alpha \in \mathbb{Z}_+^k, \quad L((\cdot)^\alpha p) = 0 \\ \Leftrightarrow & \forall \alpha \in \mathbb{Z}_+^k, \quad L(p\Pi) = 0. \end{aligned}$$

To complete the proof of (10), we need only show that the \Rightarrow above can be replaced with \Leftrightarrow .

Suppose that $L(p\Pi) = 0$. Let

$$m := \max_{x \in U} \text{totdeg } q_x$$

and fix a point $x \in U$. Choose $r \in \Pi$ so that

$$\begin{aligned} r(x) &= 1, \\ D^\gamma r(x) &= 0 \quad \text{if } 0 < |\gamma| \leq m, \text{ and} \\ D^\gamma r(v) &= 0 \quad \text{if } v \in U \setminus \{x\} \text{ and } |\gamma| \leq m. \end{aligned}$$

(For instance, one could construct r by the least interpolation scheme of Boor and Ron (1992), or more simply pick $\lambda \in \mathbb{C}^k$ for which $\lambda u \neq \lambda v$ for all $u \neq v$ in U and then construct $r = p(\lambda \cdot)$ using univariate Hermite interpolation.) Then, for every $g \in \Pi$, $v \in U$, and α of length $|\alpha| \leq m$

$$\delta_v D^\alpha(g r) = \delta_v \sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta} (D^\beta g)(D^\gamma r) = \begin{cases} 0 & \text{if } v \neq x, \text{ and} \\ D^\alpha g(x) & \text{if } v = x. \end{cases}$$

Fix $\alpha \in \mathbb{Z}_+^k$. Since $p \in I_L$,

$$0 = L((\cdot - x)^\alpha p r) = \sum_{v \in U} \delta_v q_v(D)((\cdot - x)^\alpha p r) = \delta_x q_x(D)((\cdot - x)^\alpha p),$$

which, by (11), equals $\delta_x(D^\alpha q_x)(D)p$. Since α and x were arbitrary, (10) is proven.

To show that $U = \text{var}(I_L)$, let $x \in U$ and choose a multiindex β so that $D^\beta q_x$ is a nonzero constant. Then, for any $p \in I_L$, (10) implies

$$0 = \delta_x(D^\beta q_x)(D)p,$$

so that $p(x) = 0$. Thus $U \subset \text{var}(I_L)$.

Now suppose $w \in \mathbb{C}^k \setminus U$. Pick $r \in \Pi$ so that

$$\begin{aligned} D^\alpha r(U) &= 0 \text{ if } 0 \leq |\alpha| \leq m, \text{ and} \\ r(w) &= 1. \end{aligned}$$

The first implies $r \in I_L$, and then the second implies $w \notin \text{var}(I_L)$, proving $U \supset \text{var}(I_L)$.

Finally, if q_x is a constant for every $x \in U = \text{var}(I_L)$, then

$$I_L = \{p \in \Pi : \delta_x q_x p = 0 \quad \forall x \in U\} = \{p \in \Pi : p(x) = 0 \quad \forall x \in U\} = \text{ide}(\text{var}(I_L)),$$

completing the proof. \square

The converse to Proposition 3.1 uses the following result (Lefranc, 1958), (Boor, 2005)

Theorem 3.2 (LeFranc's Nullstellensatz). *Any ideal I in $\Pi(\mathbb{C}^k)$ can be written*

$$I = \bigcap_x ({}^x \perp I) \perp_x \quad (12)$$

where, for any $S \subset \Pi$ and $x \in \mathbb{C}^k$,

$$\begin{aligned} {}^x \perp S &:= \{q \in \Pi : \forall s \in S, \delta_x q(D)s = 0\}, \text{ and} \\ S \perp_x &:= \{q \in \Pi : \forall s \in S, \delta_x s(D)q = 0\}. \end{aligned}$$

When I has finite codimension, each ${}^x \perp I$ is a finite-dimensional, D -invariant subspace of Π (Boor, 2005) and hence is nonempty only if $x \in \text{var}(I)$. Each ${}^x \perp I$ equals Π_0 exactly when $I = \text{ide}(\text{var}(I))$, i.e., I is radical.

Theorem 3.2 has been generalized to algebraically closed fields by Hakopian (2003, Theorem 5), where ${}^x \perp I$ is called the **multiplicity space** of I at x .

Proposition 3.3. *Let $\Pi = \Pi(\mathbb{C}^k)$ and suppose that $L \in \Pi'$. If I_L has finite co-dimension, then for each $x \in U := \text{var}(I_L)$ there's a unique, nonzero $q_x \in \Pi$ so that*

$$Lp = \sum_{x \in U} \delta_x q_x(D)p \quad (13)$$

for all $p \in \Pi$. If, in addition, I_L is radical, then each q_x is constant.

Proof. 3.3 Note that (10) implies that, were (13) true for some polynomials $\{q_x\}$, each q_x would necessarily be in ${}^x \perp I_L$. In particular, if I_L is radical, then each q_x would be constant.

Let

$$n := \max\{\text{totdeg } g : x \in U, g \in {}^x \perp I_L\}.$$

Then, by Boor and Ron (1992), there is a collection of polynomials

$$\{r_{x,\alpha} : x \in \text{var}(I_L), |\alpha| \leq n\}$$

with the property that

$$\delta_v D^\beta r_{x,\alpha} = \delta_{v,x} \delta_{\alpha,\beta} \quad v \in \text{var}(I_L), |\beta| \leq n.$$

Now suppose that $p \in \Pi$, and let

$$g := \sum_{x,\alpha} r_{x,\alpha} \delta_x D^\alpha p,$$

where the sum is over all $x \in U$ and $|\alpha| \leq n$, so that $\delta_x D^\alpha g = \delta_x D^\alpha p$ for all x and α . By (12), $p - g \in I_L$, and therefore

$$Lp = Lg = \sum_{x,\alpha} L(r_{x,\alpha}) \delta_x D^\alpha p = \sum_x \delta_x \sum_\alpha L(r_{x,\alpha}) D^\alpha p.$$

Letting $q_x := \sum_\alpha L(r_{x,\alpha}) ()^\alpha$ proves existence.

To show uniqueness, suppose that (13) is true when $\{q_x\}$ is replaced by some set $\{g_x\}$ of polynomials. Then, for any $v \in \text{var}(I_L)$, q_v and g_v are identical, since both are of total degree at most n , and

$$\hat{q}_v(\beta) - \hat{g}_v(\beta) = \delta_v(q_v(D) - g_v(D))r_{v,\beta} = \sum_x \delta_x(q_x(D) - g_x(D))r_{x,\beta} = 0$$

for all $|\beta| \leq n$.

Finally, to see that q_x is nonzero, let $\tilde{U} := \{x \in U : q_x \neq 0\}$. By Proposition 3.1, $\tilde{U} = \text{var}(I_L) = U$. \square

We end this section with an application of Proposition 3.3 to the solutions of multivariate difference equations.

Corollary 3.4. *Suppose J is an ideal of finite codimension in $\Pi(\mathbb{C}^k)$, and choose a monomial space B so that $\Pi = B \oplus J$. Then any sequence of complex numbers*

$$u = \{u_\alpha : ()^\alpha \in B\}$$

has a unique extension

$$v = \{v_\alpha : \alpha \in \mathbb{Z}_+^k\}$$

satisfying

$$p(E)v = 0 \quad \forall p \in J. \tag{14}$$

Furthermore, any solution to (14) is given by

$$v_\alpha = \sum_{x \in \text{var}(J)} \delta_x q_x(D) ()^\alpha$$

for some polynomials $q_x \in x \perp J$.

The existence of such a B is guaranteed by Cox et al. (1992, §3, Proposition 1 and Theorem 6).

When $k = 1$, Corollary 3.4 is a well-known result. Since all polynomial ideals in $\Pi(\mathbb{C})$ are principal, (14) can be stated simply as

$$p(E)v = 0$$

for some nonzero polynomial p . Corollary 3.4 states that, as in the univariate case, all solutions to the homogeneous difference equation (14) are polynomial in the zeros x of p and exponential polynomial in α .

Corollary 3.4 is similar to results of Hakopian and Tonoyan (2004), who prove that the solutions to linear, constant-coefficient systems of partial differential equations are exponential-polynomials obtained from the solutions to the characteristic polynomial system and their multiplicity spaces.

Proof. 3.4 Let L be the functional on B defined by $Lp := \sum_{\alpha \in B} \hat{p}(\alpha)u_\alpha$. To extend u to \mathbb{Z}_+^k so as to satisfy (14) is to extend L to Π so that $LJ = 0$. Let P be the projector from Π onto B with kernel J implied by the direct sum. Then LP is the only extension of L to Π for which $LJ = 0$, so

$$v_\alpha := LP()^\alpha \tag{15}$$

uniquely extends u to \mathbb{Z}_+^k so as to satisfy (14). This proves the first conclusion of the corollary.

Next, observe that for any polynomial p ,

$$LP((()^\alpha p) = \sum_{\gamma} (\widehat{()^\alpha p})(\gamma)LP()^\gamma = \sum_{\gamma \geq \alpha} \hat{p}(\gamma - \alpha)v_\gamma = \sum_{\beta \geq 0} \hat{p}(\beta)v_{\alpha+\beta}.$$

Consequently, $I_{LP} = \{p \in \Pi : p(E)v = 0\} \supset J$. Since J has finite codimension, so does I_{LP} . Therefore Proposition 3.3 implies that for each $x \in \text{var } I_{LP} \subset \text{var } J$ there exists a polynomial $q_x \in {}^x \perp I_{LP} \subset {}^x \perp J$ so that

$$LP = \sum_{x \in \text{var}(I_{LP})} \delta_x q_x(D)$$

which, combined with (15), completes the proof. \square

4. A generalized flat extension theorem

In this section, we see the connection between the support of a functional, the rank of its Hankel operator, and ideal projection. We also generalize an ideal projection theorem of Mourrain, another of de Boor, and the flat extension theorem of Curto and Fialkow and Laurent and Mourrain.

In any field \mathbb{F} , if B is a subspace of Π , and if $L \in (B \cdot B)'$, then the **Hankel operator** associated to L is the map

$$H^L : B \rightarrow B' : b \mapsto L(b \cdot)$$

(Laurent and Mourrain, 2009). If $B \subseteq B^+$ and $L^+ \in (B^+ \cdot B^+)'$ is an extension of L , we say H^{L^+} is a **flat extension** of H^L and write

$$H^L \xrightarrow{b} H^{L^+} \tag{16}$$

when $\text{rank } H^L = \text{rank } H^{L^+}$.

If we consider a basis \mathcal{B} for B to be a row vector of polynomials indexed by the elements of \mathcal{B} themselves, then $\mathcal{B}^T \mathcal{B}$ is a matrix of polynomials whose rows and columns are indexed by \mathcal{B} , and the representer of H^L relative to \mathcal{B} is the matrix

$$L(\mathcal{B}^T \mathcal{B}) := [L(bc)]_{b,c \in \mathcal{B}}.$$

Expand \mathcal{B} to a basis $\mathcal{B} \cup \mathcal{P}$ for B^+ , and the corresponding representer of H^{L^+} is the block matrix

$$L((\mathcal{B} \cup \mathcal{P})^T(\mathcal{B} \cup \mathcal{P})) = \begin{bmatrix} L(\mathcal{B}^T \mathcal{B}) & L(\mathcal{B}^T \mathcal{P}) \\ L(\mathcal{P}^T \mathcal{B}) & L(\mathcal{P}^T \mathcal{P}) \end{bmatrix}. \quad (17)$$

For this and $L(\mathcal{B}^T \mathcal{B})$ have the same rank, that is, for $H^L \xrightarrow{b} H^{L^+}$, is equivalent to there being a matrix A with rows and columns indexed by \mathcal{B} and \mathcal{P} so that

$$\begin{bmatrix} L(\mathcal{B}^T \mathcal{B}) & L(\mathcal{B}^T \mathcal{B})A \\ A^T L(\mathcal{B}^T \mathcal{B}) & A^T L(\mathcal{B}^T \mathcal{B})A \end{bmatrix} = \begin{bmatrix} \text{id}_{\mathcal{B}} \\ A^T \end{bmatrix} L(\mathcal{B}^T \mathcal{B}) \begin{bmatrix} \text{id}_{\mathcal{B}} & A \end{bmatrix}. \quad (18)$$

That is,

$$H^{L^+} = P' H^L P \quad (19)$$

with P the projector from B^+ onto B represented by the block matrix

$$\begin{bmatrix} \text{id} & A \end{bmatrix}$$

and P' its dual. Furthermore, if H^L is invertible, then P is uniquely determined by (19). In case $\mathbb{F} = \mathbb{R}$, (19) shows that if H^L is positive semidefinite, then so is any flat extension of H^L .

The next lemma highlights the connection between flat extensions to Π and ideal projection: the decomposition of Π as a direct sum of a finite dimensional vector space and an ideal.

Proposition 4.1. *If \mathbb{F} is any field and if $L \in \Pi'$, then the following statements are equivalent.*

- a. $\text{rank } H^L$ is finite.
- b. There exists a finite dimensional $B \subseteq \Pi$ for which $H^{L|_{B \cdot B}}$ is invertible and

$$H^{L|_{B \cdot B}} \xrightarrow{b} H^L.$$

- c. $\Pi = B \oplus I_L$ for some finite-dimensional $B \subseteq \Pi$.
- d. $L = LP$ for a projector P onto B with kernel I_L .
- e. $L = LQ$ for a finite rank ideal projector Q whose kernel lies in $\ker L$.
- f. $H^L = Q' H^{L|_{C \cdot C}} Q$ where C is the range of the finite-rank ideal projector Q .
- g. $\Pi = C \oplus J$ for some finite-dimensional subspace C and ideal $J \subset \ker L$.

Proof.

a \Rightarrow b: Choose $B \subseteq \Pi$ of dimension $\text{rank } H^L$ and bases \mathcal{B} and $\mathcal{B} \cup \mathcal{P}$ for B and Π so that

$$\begin{bmatrix} L(\mathcal{B}^T \mathcal{B}) \\ L(\mathcal{P}^T \mathcal{B}) \end{bmatrix} \quad (20)$$

and (17) have the same rank. Then (18) follows for some matrix A , so $L(\mathcal{B}^T \mathcal{B})$ has the same rank as (20) and is therefore invertible.

b \Rightarrow c: As seen in (19),

$$H^L = P' H^{L|_{B \cdot B}} P$$

for a projector P onto B . Clearly, $\ker P \subset \ker H^L$. Since the projector P is onto, its dual P' is one-to-one. This and the invertibility of $H^L|_{B \cdot B}$ imply $\ker P \supset \ker H^L$, and therefore $\ker P$ is the ideal $I_L = \ker H^L$. Consequently, $\Pi = \text{ran } P \oplus \ker P = B \oplus I_L$.

c \Rightarrow d: Let P be the ideal projection of Π onto B with kernel I_L implied by the direct sum. Then $L(\text{id} - P) = 0$ since $\ker P \subset \ker L$.

d \Rightarrow e: Trivial.

e \Rightarrow f: Since $\ker Q$ is an ideal in $\ker L$,

$$L(pq) = L(p(Qq))$$

and so, by symmetry,

$$L(pq) = L((Qp)(Qq))$$

for all p and q . (See also Boor (2005, 1.1).)

f \Rightarrow g: Let $J := \ker Q$. To see that $J \subset \ker L$, note that

$$Lq = L((Q1)(Qq)) \quad \forall q \in \Pi.$$

g \Rightarrow e \Rightarrow f \Rightarrow a: Let Q be the ideal projector from Π onto C with $\ker P = J$ guaranteed by the direct sum. Then e. and f. follow. Therefore, $\text{rank } H^L \leq \text{rank } H^L|_{C \cdot C} \leq \dim C$. \square

Theorem 4.2. $L \in \Pi(\mathbb{C}^k)'$ has the form (9) iff $\text{var}(I_L)$ is finite iff $\text{rank } H^L$ is finite. In that case, $U = \text{var}(I_L)$ and $\# \text{var}(I_L) \leq \text{rank } H^L$ with equality iff I_L is radical iff each q_x is constant.

The result $\Pi = B \oplus J$, J an ideal $\Rightarrow \# \text{var}(J) \leq \dim B$ with equality if J is radical can be found in Cox et al. (1992).

Proof. In light of Propositions 3.1, 3.3, and 4.1, all that needs to be shown is the bound on the size of $\text{var}(I_L)$ in case $\text{rank } H^L$ is finite.

Take B as in Proposition 4.1.b and P the projection of Π onto B with kernel I_L . Suppose that b and c are in B , and that $\delta_x q(D)b = \delta_x q(D)c$ for all $x \in U$ and $q \in ({}^x \perp I_L)$. Then, by (12), $b - c$ is in I_L and hence is zero. Consequently, for each $f \in \Pi$, Pf is the unique element of B which has the same values as f on the space of functionals

$$\left\{ \sum_{x \in \text{var}(I_L)} \delta_x q_x(D) : q_x \in ({}^x \perp I_L) \right\}.$$

As a result,

$$\text{rank } H^L = \dim B = \sum_x \dim({}^x \perp I_L) \geq \# \text{var}(I_L). \quad (21)$$

Being closed under differentiation, each nonempty ${}^x \perp I_L$ necessarily includes Π_0 . Therefore, equality occurs in (21) iff each ${}^x \perp I_L = \Pi_0$, which is to say, I_L is radical. \square

Recalling (7), define the monomial $(\)_j := (\)^{e_j}$. Note that $(\)_0 = 1$.

Given a projector N from B^* (8) onto $B \subseteq \Pi$, define

$$X_j : B \rightarrow B : b \mapsto N((\)_j b), \quad j = 1 : k. \quad (22)$$

If the operators X_j commute, then we can unambiguously define the operators

$$X^\alpha := \prod_{j=1}^k X_j^{\alpha(j)}$$

for any multiinteger α and

$$b(X) := \sum_{\alpha} \hat{b}(\alpha) X^\alpha \quad (23)$$

for any polynomial b .

Proposition 4.3. *If B is a finite-dimensional subspace of $\Pi(\mathbb{F}^k)$ and N is a projector from B^* onto B , then N is the restriction to B^* of a ideal projector P onto B if and only if both*

$$X_i X_j = X_j X_i \quad \forall i, j = 1 : k \quad (24)$$

and

$$b = b(X)N1 \quad \forall b \in B. \quad (25)$$

In that case, P is unique and is given by

$$Pp := p(X)N1 \quad (26)$$

Proposition 4.3 is a generalization of the earlier result Mourrain (1999, 3.1) in which B assumed to be connected to one. Condition (25) is a modification of the conclusion of Mourrain (1999, 3.2).

Proof. If $N = P|_{B^*}$ for some ideal projector, then (24) and (25) follow from Boor (2005, pp. 61-62).

Conversely, suppose (24) and (25). Then the map P given by (26) is well-defined and satisfies $P|_B = \text{id}$. Its kernel is an ideal, since

$$P(pq) = (pq)(X)N1 = q(X)p(X)N1 = q(X)P(p).$$

To see that $P|_{B^*} = N$, observe that

$$P1 = 1(X)N1 = N1$$

and that, for $b \in B$ and $j = 1 : k$,

$$P((\cdot)_j b) = (X_j b(X))N1 = X_j(b(X)N1) = X_j b = N((\cdot)_j b).$$

The uniqueness of P follows from Boor (2005, 1.7). \square

We next see that when such an N can be extended to an ideal projection P , the ideal $\ker P$ is generated by $\ker N$. This generalizes Boor (2005, 2.2), modifying the original proof to remove the requirement that $1 \in B$.

Proposition 4.4. *If P is an ideal projector with range B and $N := P|_{B^*}$, then*

$$\ker P = \text{ide}(\ker N).$$

The generating set $\{(\text{id} - N)b\}_b$, where b ranges over a basis for B^* , is called a **border basis** for $\ker P$. (Kehrein and Kreuzer, 2005), (Kehrein et al., 2005).

Proof. Since $\ker N$ lies in the ideal $\ker P$,

$$\text{ide}(\ker N) \subset \ker P.$$

Let $F := \text{span}(B \cup 1)$. Since F contains 1,

$$\Pi = \bigcup_{m \geq 0} \Pi_m \cdot F,$$

and so, to prove

$$\ker P \subset \text{ide}(\ker N),$$

it will suffice to show by induction that

$$(\Pi_m \cdot F) \cap \ker P \subset \text{ide}(\ker N) \tag{27}$$

for all nonnegative integers m .

The case $m = 0$ is straightforward:

$$(\Pi_0 \cdot F) \cap \ker P \subset B^* \cap \ker P = \ker N.$$

Assume (27) is true for some $m \geq 0$. Because F is closed under addition,

$$\Pi_{m+1} \cdot F = \Pi_1 \cdot (\Pi_m \cdot F),$$

and so, if $p \in \Pi_{m+1} \cdot F$, then there exist polynomials $q_i \in \Pi_m \cdot F$ so that

$$p = \sum_{i=0:d} ()_i q_i = \sum_{i=0:d} ()_i (Pq_i + (\text{id} - P)q_i).$$

Observe that $(\text{id} - P)q_i$ is in both $\ker P$ and $\Pi_m \cdot F + B \subset \Pi_m \cdot F$. The induction hypothesis therefore implies that $(\text{id} - P)q_i$, and therefore $()_i(\text{id} - P)q_i$, belong to $\text{ide}(\ker N)$. If p is also in $\ker P$, then $\sum_{i=0:d} ()_i Pq_i$ is in both $\ker P$ and $\Pi_1 \cdot B \subset B^*$, and hence in $\ker N$. Consequently,

$$p \in \ker N + \text{ide}(\ker N) \subset \text{ide}(\ker N),$$

completing the inductive step. \square

The following is an improvement of Laurent and Mourrain (2009, Lemma 2.1).

Proposition 4.5. *Let $L_B \in (B \cdot B)'$ for some finite dimensional $B \subseteq \Pi$ and let H^{L_B} be invertible. Suppose L_{B^*} is an extension of L_B to $(B^* \cdot B^*)'$ and that $H^{L_B} \xrightarrow{b} H^{L_{B^*}}$. Choose \mathcal{B} and $\mathcal{B} \cup \mathcal{P}$ bases for B and B^* so that*

$$L_{B^*}((\mathcal{B} \cup \mathcal{P})^T(\mathcal{B} \cup \mathcal{P})) = \begin{bmatrix} \text{id} \\ A^T \end{bmatrix} L(\mathcal{B}^T \mathcal{B}) \begin{bmatrix} \text{id} & A \end{bmatrix}$$

for some A , and let N be the projector from B^* onto B represented by the block matrix

$$\begin{bmatrix} \text{id} & A \\ 0 & 0 \end{bmatrix}.$$

Then the operators

$$X_j := N((\cdot)_j) : B \rightarrow B, \quad j = 1 : k,$$

commute.

The proof below is essentially the same as in Laurent and Mourrain (2009), except for the observation that B needn't be a monomial space containing 1.

Proof. Define

$$F := \text{id}_{B^*} - N : B^* \rightarrow B^*.$$

and

$$S_j : \Pi \rightarrow \Pi : p \mapsto (\cdot)_j p, \quad j = 1 : k,$$

so that $X_j = NS_j$ for $j = 1 : k$. Then

$$X_i X_j - X_j X_i = N(S_i NS_j - S_j NS_i) = N(S_j FS_i - S_i FS_j). \quad (28)$$

For any $q \in B^*$, because H^{LB} is invertible, Nq is the unique element of B for which $H^{LB^*} q = H^{LB^*} Nq$. Consequently, (28) equals zero iff

$$\begin{aligned} 0 &= H^{LB^*} (S_j FS_i - S_i FS_j) \\ \Leftrightarrow \forall b \in B^* \quad 0 &= L_{B^*} (b(S_j FS_i - S_i FS_j)) \\ \Leftrightarrow \forall b \in B^* \quad 0 &= L_{B^*} ((Nb)(S_j FS_i - S_i FS_j)) \\ \Leftrightarrow \forall b \in B \quad 0 &= L_{B^*} (b(S_j FS_i - S_i FS_j)) \\ &= L_{B^*} ((S_j b)FS_i - (S_i b)FS_j) \end{aligned}$$

and this last equals zero since $H^{LB^*} F = H^{LB^*} (\text{id} - N) = 0$. \square

Proposition 4.5, when combined with Propositions 4.1 and 4.3, has the following corollary.

Corollary 4.6. *Under the same hypotheses as in Proposition 4.5, L_{B^*} has an extension $L \in \Pi'$ for which*

$$H^{LB^*} \xrightarrow{b} H^L$$

iff

$$b = b(X)N1 \quad \forall b \in B, \quad (29)$$

and in that case, the extension is unique.

Proof. Suppose first that $H^{LB^*} \xrightarrow{b} H^L$. Then, by (19), there exist a projectors $Q : \Pi \rightarrow B^*$ and $N : B^* \rightarrow B$ for which

$$H^L = Q' H^{LB^*} Q = Q' N' H^{LB} N Q. \quad (30)$$

We claim that NQ is the ideal projection onto B with kernel I_L promised by Proposition 4.1. It's easy to see that it's a projection, so all that remains is to observe that, $(NQ)' : B' \rightarrow \Pi'$ is one-to-one, and being invertible, H^{LB} is also one-to-one, and so $\ker H^L = \ker(NQ)$ by (30).

Proposition 4.3 now implies (29) and that the extension of N to an ideal projection NQ is unique. By Proposition 4.1, the flat extension H^L of H^{LB^*} is also unique, completing the first half of the proof.

Conversely, suppose (25) is true. By Propositions 4.3 and 4.5, N can be uniquely extended to an ideal projector $P : \Pi \rightarrow B$. Then $H^{L_B} \xrightarrow{b} P' H^{L_B} P =: H^L$.

Since H^L , H^{L_B} , and $H^{L_{B^*}}$ all have the same rank, to prove $H^{L_{B^*}} \xrightarrow{b} H^L$, we need only show that the map $H^L : \Pi \rightarrow \Pi'$ is an extension of $H^{L_{B^*}} : B^* \rightarrow B^{*'}$. For that, note that, for any polynomials $p, q \in B^*$,

$$q' H^L p = q' P' H^{L_B} P p = q' N' H^{L_B} N p = q' H^{L_{B^*}} p.$$

□

Corollary 4.6 allows us to prove the following flat extension theorem.

Theorem 4.7. *Suppose that $B \subseteq C$ for some finite dimensional $C \subseteq \Pi$ and that $L_{C^*} \in (C^* \cdot C^*)'$. Let L_A denote the restriction of L_{C^*} to $(A \cdot A)'$ for any $A \subseteq C^*$.*

Suppose that H^{L_B} is invertible, and that

$$H^{L_B} \xrightarrow{b} H^{L_C} \xrightarrow{b} H^{L_{C^*}}$$

so that

$$H^{L_B} \xrightarrow{b} H^{L_{B^*}} \xrightarrow{b} H^{L_{C^*}}.$$

Let N and M be the associated projections from B^* and C^* onto B as in (19), and define X_i for $i = 0 : k$ as in (22).

Then L_{C^*} can be extended to $L \in \Pi'$ so that $H^{L_{C^*}} \xrightarrow{b} H^L$ iff

$$M c = c(X) N 1 \quad \forall c \in C^*. \quad (31)$$

In that case, the extension is unique.

Proof. Suppose that

$$H^{L_B} \xrightarrow{b} H^{L_{B^*}} \xrightarrow{b} H^{L_{C^*}} \xrightarrow{b} H^L.$$

Then N is extended to an ideal projection P onto B which satisfies

$$H^L = P' H^{L_B} P. \quad (32)$$

Since

$$H^{L_{C^*}} = M' H^{L_B} M, \quad (33)$$

and $H^{L_{C^*}}$ is the restriction of $H^L : \Pi \rightarrow \Pi'$ to a map $C^* \rightarrow C^{*'}$,

$$H^{L_{C^*}} = (P|_{C^*})' H^{L_B} P|_{C^*},$$

and since H^{L_B} is invertible, $M = P|_{C^*}$. Now (31) follows from (26).

Conversely, if (31) is true, then, in particular, (29) is true, so Corollary 4.6 implies that $H^{L_{B^*}}$ has a unique flat extension H^L . This by itself does not guarantee that H^L (32) is an extension of $H^{L_{C^*}}$ (33), but we prove this by noting that (31) and (26) guarantee that $P|_{C^*} = M$. □

Example 4.8. Laurent and Mourrain (2009) give the following univariate example. Let $B := \text{span}\{1, ()^3\}$, so that $B^* = \text{span}\{1, ()^1, ()^3, ()^4\}$, and choose real numbers a and b so that $b \neq a^2$. Then the rank-2 matrix

$$\begin{array}{c} 1 \quad ()^3 \\ 1 \quad \left(\begin{array}{cc} 1 & a \\ a & b \end{array} \right) \\ ()^3 \end{array} \quad (34)$$

can be flatly extended to

$$\begin{array}{c} 1 \quad ()^3 \quad ()^1 \quad ()^4 \\ 1 \quad \left(\begin{array}{cccc} 1 & a & 1 & a \\ a & b & a & b \\ ()^1 \left(\begin{array}{cccc} 1 & a & 1 & a \\ a & b & a & b \end{array} \right), \\ ()^4 \end{array} \right. \end{array} \quad (35)$$

but the latter's only possible extension to Π_4 has rank at least 3:

$$\begin{array}{c} 1 \quad ()^3 \quad ()^1 \quad ()^4 \quad ()^2 \\ 1 \quad \left(\begin{array}{ccccc} 1 & a & 1 & a & 1 \\ ()^3 \left(\begin{array}{ccccc} a & b & a & b & a \\ ()^1 \left(\begin{array}{ccccc} 1 & a & 1 & a & a \\ a & b & a & b & b \\ ()^4 \left(\begin{array}{ccccc} 1 & a & a & b & a \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \right) \end{array}.$$

From the point of view of Corollary 4.6, the reason why (35) has no flat extension is not that B fails to be connected to 1, but rather that the corresponding projection N from B^* onto B violates (29) for some b in B . Indeed, from the dependence relations in the columns of (35),

$$N : ()^1 \mapsto 1 \quad \text{and} \quad N : ()^4 \mapsto ()^3,$$

which means that $X1 = N(()^1) = 1$, and $X^3N1 = X^31 = 1 \neq ()^3$.

The matrix (34) has flat extensions to Π . For instance, when $b = -1$,

$$\begin{array}{c} 1 \quad ()^3 \quad ()^1 \quad ()^4 \\ 1 \quad \left(\begin{array}{cccc} 1 & a & -a & 1 \\ ()^3 \left(\begin{array}{cccc} a & -1 & 1 & a \\ ()^1 \left(\begin{array}{cccc} -a & 1 & -1 & -a \\ ()^4 \left(\begin{array}{cccc} 1 & a & -a & -1 \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \right) \end{array}, \quad (36)$$

is a flat extension of (34) for which the corresponding projection

$$N : ()^1 \mapsto -()^3 \quad \text{and} \quad N : ()^4 \mapsto 1.$$

satisfies $X^3N1 = ()^3$, as required. Therefore, by Theorem 4.6, (36) has a unique flat extension to Π .

In general, (34) can be extended flatly to Π . To do so, it suffices to find real numbers x_0, x_1, m_0, m_1 satisfying

$$1 = m_0 + m_1, \quad a = m_0x_0^3 + m_1x_1^3, \quad b = m_0x_0^6 + m_1x_1^6, \quad x_0 \neq x_1, \quad (37)$$

since then the moment matrix

$$[L()^{i+j}]_{i,j \geq 0}$$

of the functional

$$L : p \mapsto m_0p(x_0) + m_1p(x_1).$$

is an extension of (34) and has rank 2 by Theorem 4.2. One solution to (37) is to choose some

$$x_0 > \max \left\{ 0, \sqrt[3]{a + \sqrt{|a^2 - b|}} \right\},$$

and then define

$$x_1 := \sqrt[3]{\frac{a^2 - b}{x_0^3 - a}} + a \quad m_0 := \frac{a - x_1^3}{x_0^3 - x_1^3} \quad m_1 := \frac{x_0^3 - a}{x_0^3 - x_1^3}$$

References

- Adan, I., van Eenige, M., Resing, J., 1995. Fitting discrete distributions on the first two moments. *Probab. Engrg. Inform. Sci.* 9.04, 623–632.
- Berg, C., 1980. The multidimensional moment problem and semigroups. In: Landau, H. (Ed.), *Moments in Mathematics*. Amer. Math. Soc., Providence, RI, pp. 110–124.
- Boor, C. d., 2005. Ideal interpolation. In: Chui, C., Neamtu, M., Schumaker, L. (Eds.), *Approximation Theory XI: Gatlinburg 2004*. Nashboro Press, Brentwood TN, pp. 59–91.
- Boor, C. d., 2006. Ideal interpolation: Murrain’s condition vs. d -invariance. In: Figiel, T., Kamont, A. (Eds.), *Approximation and Probability*. Vol. 72 of Banach Center Publications. Polish Acad. Sci., Warsaw, pp. 49–55.
- Boor, C. d., Ron, A., 1991. On polynomial ideals of finite codimension with applications to box spline theory. *J. Math. Anal. Appl.* 158, 168–193.
- Boor, C. d., Ron, A., 1992. The least solution for the polynomial interpolation problem. *Math. Z.* 210.1, 347–378.
- Cools, R., 1999. Monomial cubature rules since stroud: a compilation, part 2. *J. Comput. Appl. Math.* 112.1, 21–27.
- Cools, R., Rabinowitz, P., 1993. Monomial cubature rules since stroud: a compilation. *J. Comput. Appl. Math.* 48.3, 309–326.
- Cox, D., Little, J., O’Shea, 1992. *Ideals, Varieties and Algorithms*. Undergrad. Texts Math. Springer-Verlag, New York.
- Curto, R., Fialkow, L., 1996. Solution of the truncated complex moment problem for flat data. Vol. 568 of *Mem. Amer. Math. Soc.* Amer. Math. Soc.
- Curto, R., Fialkow, L., 2000. The truncated complex k -moment problem. *Trans. Amer. Math. Soc.* 352.6, 2825–2855.
- Gasca, M., Sauer, T., 2000. Polynomial interpolation in several variables. *Adv. Comput. Math.* 12.4, 377–410.
- Hakopian, H., 2003. The multivariate fundamental theorem of algebra and algebraic geometry. In: *MEGA 2003, International Conference on Algebraic Geometry, Short Communications*. Kaiserslautern, Germany, arXiv math.AG/0403460.
- Hakopian, H., Tonoyan, M., 2004. Partial differential analogs of ordinary differential equations and systems. *New York J. Math.* 10, 89–116.
- Iokhvidov, I. S., 1982. *Hankel and Toeplitz matrices and forms: algebraic theory*. Birkhauser.
- Kehrein, A., Kreuzer, M., 2005. Characterizations of border bases. *J. Pure Appl. Algebra* 196, 251–270.

- Kehrein, A., Kreuzer, M., Robbiano, L., 2005. An algebraist's view on border bases. In: Dickenstein, A. (Ed.), *Solving Polynomial Equations: Foundations, Algorithms, and Applications*. Algorithms and Computation in Mathematics. Springer, Berlin Heidelberg, pp. 169–202.
- Lasserre, J., Laurent, M., Rostalski, P., 2009. A unified approach to computing real and complex zeros of zero-dimensional ideals. In: Putinar, M., Sullivant, S. (Eds.), *Emerging Applications of Algebraic Geometry*. The IMA Volumes in Mathematics and its Applications. Springer, New York, pp. 125–155.
- Laurent, M., 2005. Revisiting two theorems of curto and fiolkow on moment matrices. *Proc. Amer. Math. Soc.* 133, 2965–2976.
- Laurent, M., 2007. Semidefinite representations for finite varieties. *Math. Program.* 109.1, 1–26.
- Laurent, M., Mourrain, B., July 2009. A generalized flat extension theorem for moment matrices. *Arch. Math.* 93 (1), 87–98.
- Laurent, M., Rostalski, P., 2012. The approach of moments for polynomial equations. In: Anjos, M., Lasserre, J. (Eds.), *Handbook on Semidefinite, Conic and Polynomial Optimization*. International Series in Operations Research & Management Science. Springer, US, pp. 25–60.
- Lefranc, M., 1958. Analyse spectrale sur z_n . *C. R. Acad. Sci. Paris* 246, 1951–1953.
- Möller, H. M., Sauer, T., 2000a. H-bases i: The foundation. In: Cohen, A., Rabut, C., Schumaker, L. (Eds.), *Curve and Surface Fitting: Saint-Malo 1999*. Vanderbilt University Press, Nashville, TN, pp. 325–332.
- Möller, H. M., Sauer, T., 2000b. H-bases ii: applications to numerical problems. In: Cohen, A., Rabut, C., Schumaker, L. (Eds.), *Curve and Surface Fitting: Saint-Malo 1999*. Vanderbilt University Press, Nashville, TN, pp. 333–342.
- Mourrain, B., 1999. A new criterion for normal form algorithms. In: Imai, H., Lin, S., Poli, A. (Eds.), *Proceedings of the 13th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*. Vol. 1719 of Lecture Notes In Computer Science. Springer Verlag, pp. 430–443.
- Mourrain, B., Trebuchet, P., 2005. Generalized normal forms and polynomial system solving. In: Kauers, M. (Ed.), *Proceedings of the 2005 international symposium on Symbolic and algebraic computation: ISSAC '05*. New York, pp. 253–260.
- Nie, J., 2014. The a-truncated k -moment problem. *Found. Comput. Math.* 14.6, 1243–1276.
- Norton, R. M., Arnold, S., May 1985. A theorem on moments. *Amer. Statist.* 39 (2), 106.
- Pólya, G., Szego, G., 1976. *Problems and Theorems in Analysis II*. Springer Verlag, New York Heidelberg Berlin.
- Sauer, T., 1998. Polynomial interpolation of minimal degree and gröbner bases. In: Buchberger, B., Winkler, F. (Eds.), *Groebner Bases and Applications (Proc. of the Conf. 33 Years of Groebner Bases)*. Vol. 251 of London Math. Soc. Lecture Notes. Cambridge University Press, pp. 483–494.
- Sauer, T., 2001. Gröbner bases, h-bases and interpolation. *Trans. Amer. Math. Soc.* 353.6, 2293–2308.
- Shekhtman, B., 2009. Ideal interpolation: translations to and from algebraic geometry. In: Robbiano, L., Abbott, J. (Eds.), *Approximate Commutative Algebra*. Texts and Monographs in Symbolic Computation. Springer, Vienna, pp. 163–192.
- Shohat, J. A., Tamarkin, J. D., 1943. *The Problem of Moments*. American Mathematical Society, New York.