Exponential box-like splines on nonuniform grids

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In Memory of
Prof. Ewa Maria Wojcicka
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Abstract

We generalize the exponential box spline by allowing it to have arbitrarily spaced knots in any of its directions and derive the corresponding recurrence and differentiation rules. The corresponding spline space is spanned by the shifts of finitely many such splines and contains the usual family of exponential polynomials. The (local) linear independence of the spanning set is equivalent to a geometric condition closely related to unimodularity.

1. Introduction

With the invention of the exponential box spline [23], Ron generalized both the polynomial box spline of de Boor and Höllig [3] and the univariate cardinal exponential B-spline. Using the existing theory of exponential splines, one can carry this generalization a step further, replacing the uniform knot sequence of the exponential box spline with one having variable stepsize.

To see how, recall that the exponential box spline is the convolution of several simple distributions all of the form

$$\phi \mapsto \int_0^1 e^{-\mu t} \phi(\nu t)\, dt$$

for various $\mu \in \mathbb{C}$ and $\nu \in \mathbb{Z}^d$. If μ is zero, the result is the polynomial box spline. By grouping together the distributions with the same ν, one can write this as the convolution of fewer distributions, each having the form

$$\phi \mapsto \int_0^n B_\mu(t \,|\, 0, 1, \ldots, n) \phi(\nu t)\, dt$$

where $\mu \in \mathbb{C}^n$ and $\nu \in \mathbb{Z}^d$ and B_μ is a univariate exponential cardinal B-spline. If $\mu \in \mathbb{R}^n$, one can replace this cardinal spline by an exponential B-spline with arbitrarily spaced, even multiple, knots. In case the vectors $\{\nu\}$ are the standard orthonormal basis for $\mathbb{R}^d$, the resulting spline is the tensor product of the individual B-splines. As in the univariate case, splines with multiple knots are the limits of splines with coalescent knots. Figures 1.1 and 1.2 show the the supports of two bivariate splines, the first a box spline, and the second a box-like spline obtained after this replacement.

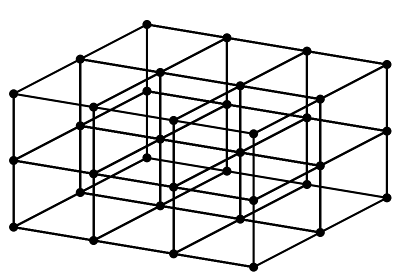

Figure 1.1: The support of a bivariate box spline.

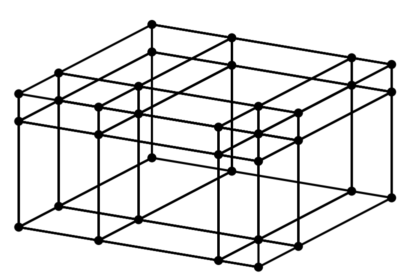

Figure 1.2: . . . and of a box-like spline.

Nonpolynomial splines have a rich history which the reader will find described elsewhere [13,27].

In its original definition and recurrence relations [23], the exponential box spline had μ real. Since then, it has always appeared with μ complex. Several fundamental properties of its integer translates were proven independently by Ron and his collaborators, Ben-Artzi and Dyn, and by Dahmen and Micchelli, including the characterization of their linear independence [9,25] and the analytic functions in their span [1,9]. Functionals dual to these translates were constructed by Jia [14], who also derived results concerning quasiinterpolants arising from, e.g., semidiscrete convolution with a compactly supported function. Jointly [5], de Boor and Ron have made a systematic study of such maps.

Dyn and Ron [11] established the approximation order of scaled exponential box spline spaces and pointed out that dilation proves to be the appropriate scaling only in the polynomial case. Earlier [10] they constructed what might best be called a Tchebycheff box spline and associated truncated Tchebycheff function. This differs from the exponential truncated power function of Dahmen and Micchelli [9]. Both of these generalize the truncated power, or cone spline, of Dahmen [7]. Sivakumar [29] allowed the exponential box spline to have rational directions $\{\nu\}$ and then investigated the linear independence of its integer translates. This is equivalent to independence of the L-translates of an exponential box spline with integral directions, where L is a sublattice of the integers. Sivakumar and Ron [26] have settled the corresponding question of approximation order in the polynomial case.

The purpose of this paper is to present similar results about the recurrence, independence, and span of the box-like spline described above. We begin by introducing some notation in Section 2. After some preliminary discussion of univariate exponential splines in Section 3, we construct the multivariate exponential spline with nonuniform knot sequences in Section 4. Even in case $\mu = 0$, this spline is not the cross-sectional area of a higher-dimensional cube; instead, see (4.4). Equation (4.2) is this spline's version of Peano's Theorem. Section 4 ends with the spline's differentiation and recursion formulas.

Section 5 deals with the span of an infinite collection of these splines and the precautions necessary to ensure that all spline series in this space converge. This space always contains a box spline space over a courser, uniform grid. Using this fact, we begin Section 6 by by finding all the exponential-polynomials in this space (Theorem 6.1). It is then proven that the global and local linear independence of the spanning set is equivalent to a simple geometric condition that, in the exponential box spline case, reduces to the unimodularity of the matrix of directions $\{\nu\}$ (Theorem 6.19). Interestingly, the necessity of this condition makes it rather difficult for the spanning set to be linearly independent (Corollary 6.7). The question then remains as to how to prescribe a basis for this space.

The polynomial version of this spline arose earlier (without multiple knots) in the study of multivariate differences and their Peano kernels [17]. Extensions of those results, of which equation (4.2) is a special case, will appear in the continuation of that work [19]. The recurrence formula for these splines in the polynomial case was proven elsewhere [18].

2. Notation

We'll let N denote a finite subset of $\mathbb{R}^d \setminus \{0\}$. One may also think of N as either a $d \times \#\mathrm{N}$ matrix or a map from $\mathbb{R}^{\mathrm{N}} := \{x : \mathrm{N} \to \mathbb{R}\}$ into $\mathbb{R}^d$ given by the usual rule

$$\mathrm{N} : x \mapsto \mathrm{N}x := \sum_{\mathrm{N}} x(\nu)\nu.$$

Its convex hull is written $\operatorname{hull}\mathrm{N}$, and the cone $\mathrm{N}[0,\infty)^{\mathrm{N}}$ is written $\operatorname{cone}\mathrm{N}$. The span of N is denoted $\operatorname{span}\mathrm{N}$. Let $\mathcal{B}_d(\mathrm{N})$ be the set of all bases for $\mathbb{R}^d$ within N. The hyperplanes spanned by N, that is, $\{\operatorname{span}\mathrm{H} : \mathrm{H} \subset \mathrm{N}, \operatorname{rank}\mathrm{H} = d-1\}$, is denoted $\mathbb{H}(\mathrm{N})$. If $x \in \mathbb{R}^{\mathrm{N}}$ and $\mathrm{H} \subset \mathrm{N}$, then x_{H} will denote the restriction of x to H. One can think of x_{H} as either a member of of $\mathbb{R}^{\mathrm{H}}$ or a member of $\mathbb{R}^{\mathrm{N}}$ whose νth coordinate is zero for all $\nu \in \mathrm{N} \setminus \mathrm{H}$.

The set of (nonnegative) multiintegers indexed by N is denoted $\mathbb{Z}^{\mathrm{N}}_{(+)}$. We will always assume that the elements of N are distinct. If $\alpha \in \mathbb{Z}^{\mathrm{N}}_{+}$, then N^α is the multiset (or $d \times |\alpha|$ matrix, or map ...) containing $\alpha(\nu)$ copies of every ν in N. We denote by i_ν the vector given by

$$i_\nu(\eta) = \begin{cases} 1 & \text{if } \nu = \eta, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

That is, i_ν is the νth column of the $\mathrm{N} \times \mathrm{N}$ identity matrix. The vector whose every coordinate is 1 is written $\mathbb{1}$.

For x and y vectors in $\mathbb{R}^{\mathrm{N}}$, we let $x < y$ and $x \le y$ have the usual meaning. Define

$$[x \,.\,.\, y] := \{u \in \mathbb{R}^{\mathrm{N}} : x \le u \le y\},$$

a box with edges parallel to the coordinate axes, possibly of zero volume. If x and y are in $\mathbb{Z}^{\mathrm{N}}$, define the discrete set

$$\{x \,.\,.\, y\} := \{u \in \mathbb{Z}^{\mathrm{N}} : x \le u \le y\}.$$

The scalar product of x and y is written $x \cdot y$. For $\theta \in \mathbb{R}^d$, the exponential function e_θ maps x to $\exp(\theta \cdot x)$.

Let D_ν denote differentiation in the direction $\nu \in \mathrm{N}$, and, for any scalar s, define $D^{\pm}_{\nu,s} := D_\nu \pm s$. More generally, if

$$\mu_\nu(1), \mu_\nu(2), \ldots, \mu_\nu(\alpha(\nu))$$

are scalars, then for any $\mathrm{H} \subseteq \mathrm{N}$ and $\beta \le \alpha$, define the two differential operators $D^{\beta,+}_{\mathrm{H},\mu}$ and $D^{\beta,-}_{\mathrm{H},\mu}$ by

$$D^{\beta,\pm}_{\mathrm{H},\mu} := \prod_{\eta \in \mathrm{H}} \prod_{i=1}^{\beta(\eta)} D_\eta \pm \mu_\eta(i).$$

When $\mu = 0$, this is written simply D^{β}_{H}.

If $F(x \,|\, P)$ denotes the value at x of a function F depending on parameters P, the function $F(\cdot \,|\, P)$ will be shortened to $F(P)$. For instance, if $S \subset \mathbb{R}^d$, then $\chi(S)$ is the function whose value $\chi(x \,|\, S)$ is 1 or 0, depending on whether or not $x \in S$.

Whenever a linear functional λ is continuous on the space of test functions, we'll make no distinction between λ and the distribution representing it.

Since a vector, say $x \in \mathbb{R}^{\mathrm{N}}$, is simply a function with finite domain, $\operatorname{supp} x$ is the set of those $\nu \in \mathrm{N}$ for which $x(\nu) \ne 0$.

3. Univariate Exponential B-splines

In this section we briefly review some known facts about univariate exponential splines. Two books [15, 27] together make an ample reference for this material.

Given the real numbers $\mu_1, \ldots, \mu_n$, let $\mathrm{EXP}(\mu_1, \ldots, \mu_n)$ denote the univariate function space spanned by $\{()^{m_j} e_{\mu_j}\}_{j=1}^n$, where $m_j = \#\{i < j : \mu_i = \mu_j\}$ and $()^m$ is the mth degree monomial.

Let f be a function given at the real numbers $t_0 \le \cdots \le t_n$. For any real number a, natural number k, and integer i satisfying $0 \le i \le n-k$, there exists a unique function p in $\mathrm{EXP}(\mu_1, \ldots, \mu_k, a)$ such that $f - p$ vanishes at $t_i, \ldots, t_{i+k}$, counted according to their multiplicities. Let m be the multiplicity of a in the list $\mu_1, \ldots, \mu_k, a$. Define the **divided difference** of f (at $t_i, \ldots, t_{i+k}$, and with respect to $\mu_1, \ldots, \mu_k$ and a)

$$\begin{bmatrix} t_i, \ldots, t_{i+k} \\ \mu_1, \ldots, \mu_k, a \end{bmatrix} f$$

to be the scalar c such that $p - c()^{m-1} e_a$ is an element of $\mathrm{EXP}(\mu_1, \ldots, \mu_k)$. By letting ";" indicate a carriage return, the divided difference functional

$$\begin{bmatrix} t_i, \ldots, t_{i+k} \\ \mu_1, \ldots, \mu_k, a \end{bmatrix}$$

may be written in a single line as $[t_i, \ldots, t_{i+k}; \mu_1, \ldots, \mu_k, a]$. From its definition, the divided difference is symmetric in both $t_i, \ldots, t_{i+k}$ and $\mu_1, \ldots, \mu_k$. Since this functional annihilates $\mathrm{EXP}(\mu_1, \ldots, \mu_k)$, and this space has dimension k over $\{t_i, \ldots, t_{i+k}\}$, there is, up to a scalar multiple, only one such functional. Consequently [20],

$$\begin{bmatrix} t_0, \ldots, t_{n-1} \\ \mu_1, \ldots, \mu_n \end{bmatrix} = \begin{bmatrix} t_0, \ldots, t_{n-1} \\ \mu_1, \ldots, \mu_n \end{bmatrix} u_0^- \begin{bmatrix} t_0, \ldots, t_{n-1} \\ \mu_1, \ldots, \mu_{n-1}, 0 \end{bmatrix} \tag{3.1}$$

and

$$\begin{bmatrix} t_0,\dots,t_n \\ \mu_1,\dots,\mu_n,0 \end{bmatrix} = \frac{\begin{bmatrix} t_0,\dots,t_{n-1} \\ \mu_1,\dots,\mu_n \end{bmatrix} - \begin{bmatrix} t_1,\dots,t_n \\ \mu_1,\dots,\mu_n \end{bmatrix}}{\left(\begin{bmatrix} t_0,\dots,t_{n-1} \\ \mu_1,\dots,\mu_n \end{bmatrix} - \begin{bmatrix} t_1,\dots,t_n \\ \mu_1,\dots,\mu_n \end{bmatrix}\right) u_0^+} \qquad (t_0 \neq t_n) \tag{3.2}$$

where u_0^+ and u_0^- are the highest degreed monomials in $\mathrm{EXP}(\mu_1,\dots,\mu_n,0)$ and $\mathrm{EXP}(\mu_1,\dots,\mu_{n-1},0)$, respectively. The denominator in (3.2) is zero if and only if $t_0 = t_n$, and is sometime written as a quotient of determinants [6].

By mimicking the case $\mu_1 = \cdots = \mu_n = 0$, when $[t_0,\dots,t_n;\mu_1,\dots,\mu_n,0]$ is an ordinary divided difference, one can prove the following results.

Theorem 3.3. Generalized Rolle's Theorem. *If the real-valued function f is continuous on $[a,b]$ and differentiable on (a,b), if $f(a) = f(b) = 0$, and if μ is any real number, then there exists a point ξ strictly between a and b such $(D-\mu)f(\xi) = 0$.*

Corollary 3.4. *If $t_0 \le \cdots \le t_n$ and $\mu_1,\dots,\mu_n$ are real, if $D^n f$ exists on (t_0,t_n), and if $D^k f$ is continuous $(0 \le k < n)$ on $[t_0,t_n]$, then there is a point ξ in $[t_0,t_n]$ at which*

$$\begin{bmatrix} t_0,\dots,t_n \\ \mu_1,\dots,\mu_n,0 \end{bmatrix} f = \frac{\prod_{i=1}^n (D-\mu_i)f(\xi)}{m!\prod'(-\mu_i)}$$

where m is the multiplicity of 0 in $\mu_1,\dots,\mu_n$ and $\prod'$ is the product over those i in $\{1\,..\,n\}$ for which μ_i is nonzero.

It is easy to see that, for any real numbers a and q and function f,

$$\begin{bmatrix} t_0+q,\dots,t_n+q \\ \mu_1,\dots,\mu_n,a \end{bmatrix} f = e^{-aq} \begin{bmatrix} t_0,\dots,t_n \\ \mu_1,\dots,\mu_n,a \end{bmatrix} f(\cdot + q). \tag{3.5}$$

Let $\mu := \{\mu_1,\dots,\mu_n\} \subset \mathbb{R}$ and define a distribution T_μ by its action on test functions:

$$\langle T_\mu, \phi \rangle := \int_{x\in[0,\infty)^n} e^{-\mu\cdot x} \phi(x(1)+\cdots+x(n))\,dx.$$

T_μ is just a special case of more general exponential or Tchebycheff "truncated powers" [9,12,16]. Its support is $[0,\infty)$, and its restriction to $(0,\infty)$ is in $\mathrm{EXP}(-\mu_1,\dots,-\mu_n)$. Some of its important known properties are that T_μ is symmetric in $\mu_1,\dots,\mu_n$, that

$$T_{\mu_1} * T_{\mu_2,\dots,\mu_n} = T_{\mu_1,\dots,\mu_n},$$

and that

$$(D+\mu_1)T_{\mu_1,\dots,\mu_n} = T_{\mu_2,\dots,\mu_n}. \tag{3.6}$$

Moreover, T_μ is $n-2$ times continuously differentiable and $\prod_{i=1}^n (D+\mu_i)T_\mu$ is the Dirac δ.

The exponential B-spline corresponding to $\mu = \{\mu_1,\dots,\mu_n\}$ and the knots $t_0 \le \cdots \le t_n$ is defined as the convolution

$$B_\mu(t_0,\dots,t_n) := \begin{bmatrix} t_0,\dots,t_n \\ \mu_1,\dots,\mu_n,0 \end{bmatrix} * T_\mu. \tag{3.7}$$

Since T_μ is piecewise in $\mathrm{EXP}(-\mu_1,\dots,-\mu_n)$, so is B_μ. Equation (3.5) implies that, for any real number q,

$$B_\mu(\cdot + q \,|\, t_0,\dots,t_n) = B_\mu(\cdot \,|\, t_0+q,\dots,t_n+q). \tag{3.8}$$

That is, to shift the knots is to shift the spline. Like the classical B-spline, B_μ is supported on $[t_0\,..\,t_n]$, since the restriction of $T_\mu(x-\cdot)$ to either $(-\infty,x)$ or (x,∞) is in $\mathrm{EXP}(\mu_1,\dots,\mu_n)$. Furthermore, the support of B_μ is minimal in the following sense.

Lemma 3.9. *Let $t_0 \le \cdots \le t_{n-1}$ and $\mu = \{\mu_1, \ldots, \mu_n\} \subset \mathbb{R}$. Define $m_j = \#\{i < j : t_i = t_j\}$. If $a_0, \ldots, a_{n-1}$ are real numbers with the property that*

$$\sum_{j=0}^{n-1} a_j D^{m_j} T_\mu(\cdot - t_j)$$

has compact support, then $a_0 = \cdots = a_{n-1} = 0$.

Corollary 3.10. *If $[t_0, \ldots, t_n; \mu_1, \ldots, \mu_n, 0]$ is given by the rule*

$$f \mapsto \sum_{j=0}^{n} a_j D^{m_j} f(t_j),$$

then a_n is nonzero. Furthermore, if $t_0 = \cdots = t_k < t_{k+1}$, then a_k is nonzero.

The standard derivative formula for the exponential B-spline follows from identities (3.1), (3.2), and (3.6):

$$(D+\mu_n)B_{\mu_1,\ldots,\mu_n}(t_0, \ldots, t_n) = \tag{3.11}$$
$$\frac{B_{\mu_1,\ldots,\mu_{n-1}}(t_0, \ldots, t_{n-1}) \begin{bmatrix} t_0, \ldots, t_{n-1} \\ \mu_1, \ldots, \mu_n, \end{bmatrix} u_0^- - B_{\mu_1,\ldots,\mu_{n-1}}(t_1, \ldots, t_n) \begin{bmatrix} t_1, \ldots, t_n \\ \mu_1, \ldots, \mu_n, \end{bmatrix} u_0^-}{\begin{bmatrix} t_0, \ldots, t_{n-1} \\ \mu_1, \ldots, \mu_n, \end{bmatrix} u_0^+ - \begin{bmatrix} t_1, \ldots, t_n \\ \mu_1, \ldots, \mu_n, \end{bmatrix} u_0^+}$$

where u_0^+ and u_0^- are the highest degreed monomials in $\mathrm{EXP}(\mu_1, \ldots, \mu_n, 0)$ and $\mathrm{EXP}(\mu_1, \ldots, \mu_{n-1}, 0)$, respectively.

Lemma 3.12. *Let $t_0 \le \cdots \le t_k$ with $k \ge n$. Assume further that $t_i < t_{i+n-1}$ for all i. Let λ be a linear functional of the form*

$$\lambda : f \mapsto \sum_{i=0}^{k} a_i D^{m_i} f(t_i)$$

where $m_i := \#\{j < i : t_j = t_i\}$. If λ annihilates $\mathrm{EXP}(\mu_1, \ldots, \mu_n)$, then λ can be written as a linear combination of the divided differences

$$\left\{ \begin{bmatrix} t_i, \ldots, t_{i+n} \\ \mu_1, \ldots, \mu_n, 0 \end{bmatrix} : i \in \{0\,.\,.\,k-n\} \right\}.$$

Consequently, any exponential B-spline B_μ with $n+1$ knots chosen from $\{t_0, \ldots, t_k\}$ is a linear combination of the splines

$$\{B_\mu(t_i, \ldots, t_{i+n}) : i \in \{0\,.\,.\,k-n\}\}.$$

By (3.7), $\prod_{i=1}^n (D+\mu_i) B_\mu(t_0, \ldots, t_n)$ and $[t_0, \ldots, t_n; \mu_1, \ldots, \mu_n, 0]$ are the same distribution. Hence the Peano kernel relation is

$$(-1)^n \begin{bmatrix} t_0, \ldots, t_n \\ \mu_1, \ldots, \mu_n, 0 \end{bmatrix} f = \int_{t_0}^{t_n} B_\mu(x \,|\, t_0, \ldots, t_n) \prod_{i=1}^{n} (D - \mu_i) f(x)\, dx. \tag{3.13}$$

for any sufficiently differentiable function f. Consequently, Corollary 3.4 implies that, except when it is zero, $B_\mu(t_0, \ldots, t_n)$ has the same sign as $(-1)^n \prod'(-\mu_i)$, where $\prod'$ is the product over those i between 1 and n for which μ_i is nonzero.

We finish this section by mentioning two other consequences of equation (3.13).

First, for any real number m,

$$\int_{-\infty}^{\infty} e^{mx} B_\mu(x \,|\, t_0, \ldots, t_n)\, dx \ne 0. \tag{3.14}$$

Second, if $\mu_1 = \cdots = \mu_n =: \mu$, then, up to a constant factor, B_μ equals the polynomial B-spline B_0 times the exponential function $e_{-\mu}$. Specifically,

$$B_\mu(t_0, \ldots, t_n)[t_0, \ldots, t_n]e_{-\mu} = e_{-\mu}B_0(t_0, \ldots, t_n), \tag{3.15}$$

as can be seen by integrating each side against $(D-\mu)^n(e_\mu f)$ for f an arbitrary test function.

4. A multivariate exponential box-like spline

Assume that no two elements of N are parallel, and that $0 \notin \mathrm{N}$. Assume also that $\alpha \in \mathbb{Z}_+^{\mathrm{N}}$ and that $\mu \in \mathbb{R}^{\mathrm{N}^\alpha}$, or, equivalently, that $\mu_\nu(i)$ is a real number for every $\nu \in \mathrm{N}$ and $i \in \{1\,..\,\alpha(\nu)\}$.

Define a map $\tau : \mathbb{Z}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$ as follows. For each $\nu \in \mathrm{N}$, let $\{t_\nu(n) : n \in \mathbb{Z}\}$ be a nondecreasing biinfinite sequence of real numbers with $t_\nu(0) = 0$. For $z \in \mathbb{Z}^{\mathrm{N}}$, define $\tau(z)$ to be the vector in $\mathbb{R}^{\mathrm{N}}$ whose νth coordinate is $\tau_\nu(z) := t_\nu(z(\nu))$. Either τ or t_ν may be called a **knot sequence**. For H a subset of N, identify $\mathbb{R}^{\mathrm{H}}$ (resp. $\mathbb{Z}^{\mathrm{H}}$) with the subset of $\mathbb{R}^{\mathrm{N}}$ (resp. $\mathbb{Z}^{\mathrm{N}}$) consisting of those vectors supported on H. Since each $t_\nu(0) = 0$, if $z \in \mathbb{Z}^{\mathrm{H}}$, then $\tau(z) \in \mathbb{R}^{\mathrm{H}}$.

Definition 4.1. *Given a directional matrix* N*, a vector* $\mu \in \mathbb{R}^{\mathrm{N}^\alpha}$*, and knot sequence* τ*, define the d-variate distribution* $B_\mu(z, \tau, \mathrm{N}^\alpha)$ *to be the convolution of the collection of distributions*

$$\left\{ f \mapsto \int_{-\infty}^{\infty} f(x\nu) B_{\mu_\nu}\big(x \,|\, \tau_\nu(z), \ldots, \tau_\nu(z+\alpha)\big)\, dx \;:\; \nu \in \mathrm{N} \right\}.$$

The support of $B_\mu(z, \tau, \mathrm{N}^\alpha)$ is the image in $\mathbb{R}^d$ under N of the box $[\tau(z)\,..\,\tau(z+\alpha)]$ in $\mathbb{R}^{\mathrm{N}}$. On its support, $B_\mu(z, \tau, \mathrm{N}^\alpha)$ has the same sign as $(-1)^{|\alpha|}\prod'(-\mu_\nu(j))$, where $\prod'$ is the product over the members of N^α for which $\mu_\nu(j)$ is nonzero.

It follows from equation (3.13) that, for any test function ϕ,

$$(-1)^{|\alpha|}\delta^\alpha_{z,\mu}\phi = \langle B_\mu(z, \tau, \mathrm{N}^\alpha), D^-_{\mathrm{N}^\alpha,\mu}\phi\rangle \tag{4.2}$$

where $\delta^\alpha_{z,\mu}$ is the convolution of the collection of functionals

$$\left\{ f \mapsto \begin{bmatrix} \tau_\nu(z), \ldots, \tau_\nu(z+\alpha) \\ \mu_\nu(1), \ldots, \mu_\nu(\alpha(\nu)), 0 \end{bmatrix} f(\nu\cdot) \;:\; \nu \in \mathrm{N} \right\}.$$

If $\tau(z) = z$ for all $z \in \mathbb{Z}^{\mathrm{N}}$, then $B_\mu(z, \tau, \mathrm{N}^\alpha)$ is, up to a scalar multiple, a shift of the exponential box spline $C_\mu(\mathrm{N}^\alpha)$, the distribution defined [23] by the rule

$$\langle C_\mu(\mathrm{N}^\alpha), \phi\rangle := \int_{[0..1]^{\mathrm{N}^\alpha}} e^{-\mu\cdot y}\phi(\mathrm{N}^\alpha y)\, dy, \tag{4.3}$$

since, in this case, $\delta^\alpha_{z,\mu}$ is a constant times the exponential forward difference functional. More generally, if $\tau(z)$ equals qz for all z in $\mathbb{Z}^{\mathrm{N}}$ and some scalar q, then $B_\mu(z, \tau, \mathrm{N}^\alpha)$ is a constant times $C_{q\mu}(q\mathrm{N}^\alpha)$. The closest thing to equation (4.3) that holds for all splines $B_\mu(z, \tau, \mathrm{N}^\alpha)$ seems to be equation (4.4) below.

Define the function $W_\mu(z, \tau, \alpha)$ on $\mathbb{R}^{\mathrm{N}}$ to be the tensor product of the univariate B-splines

$$\{B_{\mu_\nu}(\tau_\nu(z), \ldots, \tau_\nu(z+\alpha)) : \nu \in \mathrm{N}\}.$$

Then, for any d-variate test function ϕ,

$$\langle B_\mu(z, \tau, \mathrm{N}^\alpha), \phi\rangle = \langle W_\mu(z, \tau, \alpha), \phi \circ \mathrm{N}\rangle. \tag{4.4}$$

That is, $B_\mu(x \,|\, z, \tau, \mathrm{N}^\alpha)$ is the integral of the weight function W over a cross section of the box $[\tau(z)\,..\,\tau(z+\alpha)]$ in $\mathbb{R}^{\mathrm{N}}$.

The derivative formula for $B_\mu(z,\tau,\mathrm{N}^\alpha)$ follows from Definition 4.1 and the univariate identity (3.11). If $\nu \in \mathrm{N}$ and $j \in \{1\,..\,\alpha(\nu)\}$, and if $\tau_\nu(z) \neq \tau_\nu(z+\alpha)$, then

$$\big(D_\nu + \mu_\nu(j)\big)B_\mu(z,\tau,\mathrm{N}^\alpha) = \frac{s^-(z)B_\mu(z,\tau,\mathrm{N}^{\alpha-i_\nu}) - s^-(z+i_\nu)B_\mu(z+i_\nu,\tau,\mathrm{N}^{\alpha-i_\nu})}{s^+(z)-s^+(z+i_\nu)} \tag{4.5}$$

where

$$s^\pm(z) = \begin{bmatrix} \tau_\nu(z),\ldots,t_\nu(z+\alpha-i_\nu) \\ \mu_\nu(\{1\,..\,\not{j}\,..\,\alpha(\nu)\}),\mu_\nu(j) \end{bmatrix} u_\nu^\pm, \tag{4.6}$$

where u_ν^+ is the highest-degree monomial in the space of unvariate functions $\mathrm{EXP}(\mu_\nu(\{1\,..\,\alpha(\nu)\}),0)$, and where u_ν^- is the highest monomial in $\mathrm{EXP}(\mu_\nu(\{1\,..\,\not{j}\,..\,\alpha(\nu)\}),0)$.

When $\mu \equiv 0$, the monomials $u_\nu^\pm$ are of degree $\alpha(\nu) - \frac{1}{2} \pm \frac{1}{2}$ and (4.5) simplifies to

$$D_\nu B_0(z,\tau,\mathrm{N}^\alpha) = \frac{B_0(z+i_\nu,\tau,\mathrm{N}^{\alpha-i_\nu}) - B_0(z,\tau,\mathrm{N}^{\alpha-i_\nu})}{\tau_\nu(z+\alpha)-\tau_\nu(z)},$$

an extension of the familiar B-spline formula [2].

Define the space $D_\mu(\mathrm{N}^\alpha)$ of d-variate distributions to be the intersection of the kernels of $D^+_{\mathrm{M},\mu}$ for all $\mathrm{M} \subseteq \mathrm{N}^\alpha$ for which $\mathrm{N}^\alpha \setminus M$ does not span $\mathbb{R}^d$. It is known [1,9,14] that $D_\mu(\mathrm{N}^\alpha)$ is spanned by exponential-polynomials and has dimension equal to the number of bases for $\mathbb{R}^d$ found in N^α. Since $B_\mu(z,\tau,\mathrm{M})$ is singular if M does not span $\mathbb{R}^d$, it follows from equation (4.5) that $B_\mu(z,\tau,\mathrm{N}^\alpha)$ is piecewise in $D_\mu(\mathrm{N}^\alpha)$.

The pair (N^α,μ) is said to be **single-noded** [23] if there exists a $\theta \in \mathbb{R}^d$ such that $\mu = \theta^T\mathrm{N}^\alpha$; that is, $\mu_\nu(i) = \theta\cdot\nu$ for every $\nu \in \mathrm{N}$ and $i \in \{1\,..\,\alpha(\nu)\}$. As in the univariate case if (N^α,μ) is single-noded, $B_\mu(z,\tau,\mathrm{N}^\alpha)$ is a scalar multiple of $e_{-\theta}B_0(z,\mathrm{N}^\alpha)$. We omit the proof of this simple result, which follows directly from equation (3.15) and the definition of $B_\mu(z,\tau,\mathrm{N}^\alpha)$.

Lemma 4.7. *Under the same hypotheses as Definition 4.1, if $\mu = \theta^T\mathrm{N}^\alpha$ for some $\theta \in \mathbb{R}^d$, then*

$$B_\mu(z,\tau,\mathrm{N}^\alpha)\Lambda^\alpha_{z,\theta} = e_{-\theta}B_0(z,\tau,\mathrm{N}^\alpha)$$

where

$$\Lambda^\alpha_{z,\theta} = \prod_{\substack{\nu\in\mathrm{N} \\ \theta\cdot\nu\neq 0}} [\tau_\nu(z),\ldots,\tau_\nu(z+\alpha)]e_{-\theta\cdot\nu}.$$

One consequence of Lemma 4.7 is that the recursion formula for $B_\mu(z,\tau,\mathrm{N}^\alpha)$ in the single-noded case is nearly the same as in the case $\mu \equiv 0$. We next state the recursion formula in general.

Theorem 4.8. *Let N^α, μ, and τ be as in Definition 4.1. Let $\tau_\nu(z) < \tau_\nu(z+\alpha)$ for all ν in N.*

a. *If (N^α,μ) is single-noded with $\mu = \theta^T\mathrm{N}^\alpha$, and if $x = \mathrm{N}\xi$ for some $\xi \in \mathbb{R}^{\mathrm{N}}$, then*

$$\begin{aligned}(|\alpha|-d)B_\mu(x\,|\,z,\tau,\mathrm{N}^\alpha)\Lambda^\alpha_{z,\theta} = \sum_{\nu\in\mathrm{N}} &(w_{z,\nu}(\xi(\nu))-1)B_\mu(x\,|\,z+i_\nu,\tau,\mathrm{N}^{\alpha-i_\nu})\Lambda^{\alpha-i_\nu}_{z+i_\nu,\theta} \\ &- w_{z,\nu}(\xi(\nu))B_\mu(x\,|\,z,\tau,\mathrm{N}^{\alpha-i_\nu})\Lambda^{\alpha-i_\nu}_{z,\theta},\end{aligned}$$

with Λ as in Lemma 4.7, and

$$w_{z,\nu}(\cdot) = \frac{\cdot - \tau_\nu(z)}{\tau_\nu(z+\alpha)-\tau_\nu(z)}.$$

b. *If (N^α,μ) is not single-noded, then*

$$B_\mu(z,\tau,\mathrm{N}^\alpha) = \sum_{\nu\in\mathrm{N}}\sum_{j=1}^{\alpha(\nu)} b_\nu(j)\frac{s^-(z)B_{\mu'}(z,\tau,\mathrm{N}^{\alpha-i_\nu}) - s^-(z+i_\nu)B_{\mu'}(z+i_\nu,\tau,\mathrm{N}^{\alpha-i_\nu})}{s^+(z)-s^+(z+i_\nu)}$$

where $\mu' = \mu \setminus \mu_\nu(j)$, the functions $s^\pm$ are defined in (4.6), and $b \in \mathbb{R}^{\mathrm{N}^\alpha}$ is chosen to satisfy $\mathrm{N}^\alpha b = 0$ and $\mu \cdot b = 1$.

Proof of part a.: The case $\mu = 0$ was proven earlier [18] and the general case follows from this via Lemma 4.7. ■

Proof of part b.: The existence of a b in $\mathbb{R}^{\mathrm{N}^\alpha}$ satisfying $\mathrm{N}^\alpha b = 0$ and $\mu \cdot b = 1$ is a consequence of (N^α, μ) not being single-noded. The recursion follows from

$$B_\mu(z, \tau, \mathrm{N}^\alpha) = (D_{\mathrm{N}^\alpha b} + \mu \cdot b) B_\mu(z, \tau, \mathrm{N}^\alpha)$$

and formula (4.5). ■

The proof of part a. in case $\mu = 0$ is a modification of the proof of the (polynomial) box spline's recursion [4]. The technique used to extend this result to the case $\mu \neq 0$, as well as that used to prove part b., are taken from Ron's original paper [23].

5. The spline space S

In this section we assume N has rank d and consider the space

$$S := \operatorname{span} B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$$

where

$$B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha) := \{B_\mu(z, \tau, \mathrm{N}^\alpha) : z \in \mathbb{Z}^{\mathrm{N}}\}.$$

Define the set of shifted hyperplanes

$$\begin{aligned}\Gamma(\mathrm{N}, \tau) &:= \mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}}) + \mathbb{H}(\mathrm{N}) \\ &= \{\mathrm{N}\tau(z) + \operatorname{span} \mathrm{H} : z \in \mathbb{Z}^{\mathrm{N}},\ \mathrm{H} \subset \mathrm{N},\ \operatorname{rank} \mathrm{H} = d - 1\}\end{aligned}$$

where $\mathrm{N}\tau(z) = \sum_{\mathrm{N}} \tau_\nu(z)\nu$. On any domain not intersecting the hyperplanes in $\Gamma(\mathrm{N}, \tau)$, the elements of S belong to $D_\mu(\mathrm{N}^\alpha)$.

Note that, even if $\tau(z) = z$ for all $z \in \mathbb{Z}^{\mathrm{N}}$, in which case each $B_\mu(z, \tau, \mathrm{N}^\alpha)$ is a multiple and shift of the box spline $C_\mu(\mathrm{N}^\alpha)$, the spanning set $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$ is different from the one usually studied in the box spline literature. In that case, $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$ consists, up to a constant factor, of all shifts of $C_\mu(\mathrm{N}^\alpha)$ by elements in $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$, a possibly proper sublattice of the integers $\mathbb{Z}^d$. The motivation for studying something other than the usual spanning set was the observation that the splines

$$\{C_\mu(\cdot - z \,|\, \mathrm{N}^\alpha) : z \in \mathbb{Z}^d\}. \tag{5.1}$$

must be linearly dependent if $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$ is a proper subset of $\mathbb{Z}^d$.

For arbitrary N and τ, it is possible for the supports of infinitely many splines in S to overlap, as the following example shows.

Example 5.2. $\mathrm{N} = \begin{pmatrix} 1 & 0 & \pi \\ 0 & 1 & 1 \end{pmatrix}$, $\alpha = \mathbb{1} = (1, 1, 1)$, $\mu = 0$, and $\tau(z) = z$ for all $z \in \mathbb{Z}^{\mathrm{N}}$. Since $\mathbb{Z} + \pi\mathbb{Z}$ is dense on the real line, it follows that, for any integer n, the points $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$ are dense along the line $\{(x, n) : x \in \mathbb{R}\}$. Since the support of $B_0(z, \tau, \mathrm{N}^{\mathbb{1}})$ is $\mathrm{N}\tau(z) + \mathrm{N}[0\,..\,1]^3$, at any point (x, y) in $\mathbb{R}^2$ with noninteger y, there exist infinitely many nonzero splines $B_0(z, \tau, \mathrm{N}^{\mathbb{1}})$.

By restricting N and τ to contain only integers, one can avoid the density of the grid lines, but the supports of infinitely many splines may still intersect, as we see next.

Example 5.3. $\mathrm{N} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\alpha = \mathbb{1}$, $\mu = 0$, and

$$\tau(z) = \begin{cases} z & \text{if } z(3) \le 0, \text{ and} \\ (z(1), z(2), 2^{z(3)}) & \text{if } z(3) > 0. \end{cases}$$

Then $B_0(z, \tau, \mathrm{N}^{\mathbb{1}})$ is nonzero at the point $(\frac{1}{2}, \frac{1}{2})$ for any z of the form $(-2^n, -2^n, n)$ for n a natural number.

The overlapping supports in these examples would mean that spline series of the form

$$\sum_{\mathbb{Z}^{\mathrm{N}}} a_z B_\mu(x \,|\, z, \tau, \mathrm{N}^\alpha)$$

would diverge, at least at some points $x \in \mathbb{R}^d$, without some conditions placed on the coefficients $\{a_z\}$.

In order for S to be locally finite dimensional, one apparently needs to impose some order on the knots $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$. As we'll see by the end of this section, the following four assumptions will suffice without requiring that the knots be unformly spaced.

Assumption 5.4. N *is a integer matrix with rank* d. *No two of its elements are parallel,* hull N *does not contain the origin, and* α *is a positive element of* $\mathbb{Z}_+^{\mathrm{N}}$.

Assumption 5.5. *Each* t_ν *is a nonconstant, nondecreasing sequence of rational numbers with* $t_\nu(0) = 0$ *and periodic stepsize. That is, there exists a natural number* $p(\nu)$ *such that for any* n *in* $\mathbb{Z}$,

$$\Delta t_\nu(n + p(\nu)) = \Delta t_\nu(n) := t_\nu(n+1) - t_\nu(n).$$

Define the multiindex p to have $p(\nu)$ for its νth coordinate. If $z \in \mathbb{Z}^{\mathrm{N}}$, we say that $p \,|\, z$ if $p(\nu)$ divides $z(\nu)$ for every ν.

Assumption 5.6. *If* $\mathrm{N}\tau(u) = \mathrm{N}\tau(w)$ *for some* u *and* w *in* $\mathbb{Z}^{\mathrm{N}}$, *then there exists* $u' \in \mathbb{Z}^{\mathrm{N}}$ *with* $\tau(u) = \tau(u')$ *and* $p \,|\, (w - u')$.

Assumption 5.7. *If* $\nu \in \mathrm{N}$ *and* $z \in \mathbb{Z}^{\mathrm{N}}$, *then* $\tau_\nu(z) < \tau_\nu(z + \alpha)$.

Allowing t_ν to be rational means that knots could coalesce in some or all of the (integral) directions N, without necessarily violating the Assumptions 5.4–5.7.

In case each t_ν is strictly increasing, Assumption 5.6 states that no two knots $\mathrm{N}\tau(u)$ and $\mathrm{N}\tau(w)$ can be equal unless $p \,|\, (w - u)$. In that case, if some points of τ coalesce (without changing the period p of the stepsize), the limiting knot sequence will still satisfy Assumption 5.6.

Assumption 5.7 is equivalent to the univariate spline

$$B_{\mu_\nu}(\tau_\nu(z), \ldots, \tau_\nu(z + \alpha))$$

having support at more than just a single point.

Example 5.8. Assumptions 5.4–5.7 are satisfied by the matrix $\mathrm{N} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and knot sequence τ consisting of the sequences

$$t_{\binom{1}{0}} = t_{\binom{0}{1}} = \{\ldots, -4, -3, 0, 1, 4, 5, \ldots\}$$
$$t_{\binom{1}{1}} = \{\ldots, -4, -2, -2+\epsilon, 0, 2, 2+\epsilon, 4, 6, 6+\epsilon, \ldots\}$$

where ϵ is a rational number between 0 and 1. Here, $p = (2, 2, 3)$.

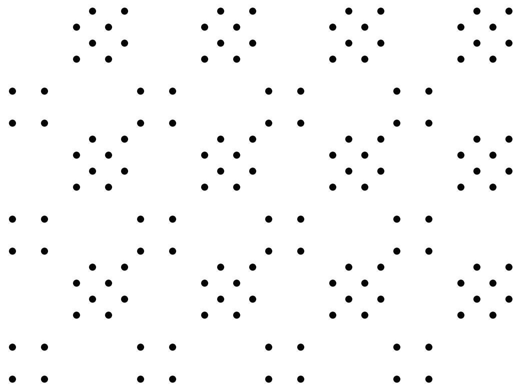

Figure 5.9: $\epsilon = \frac{1}{2}$

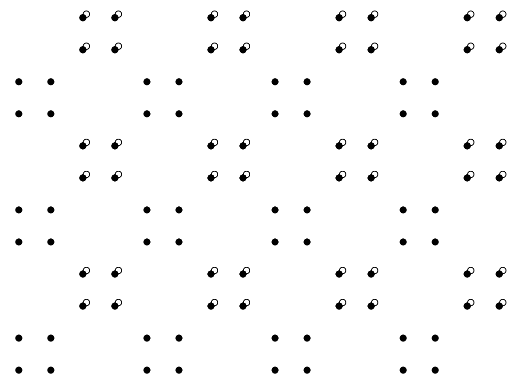

Figure 5.10: $\epsilon = 0$

Figure 5.9 shows the knots $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$ of Example 5.8 when $\epsilon = 1/2$. The point in the lower left corner is the origin. In Figure 5.10, ϵ is zero. The double knots are marked with an additional $\circ$, placed off-center as a reminder of the direction in which the multiplicity occurs.

The next lemma is a useful (and immediate) consequence of Assumption 5.5.

Lemma 5.11. *Under Assumptions 5.4–5.7,*

$$t_\nu(mp(\nu) + n) = mt_\nu(p(\nu)) + t_\nu(n)$$

for any ν in N *and integers m and n.*

We note two consequences of Lemma 5.11. The first also uses our assumptions that t_ν is nondecreasing and nonconstant.

Corollary 5.12. *For any ν in* N, $t_\nu(p(\nu)) \neq 0$.

Corollary 5.13. *If u and w in $\mathbb{Z}^{\mathrm{N}}$ satisfy $p \,|\, (u - w)$, then $\tau(u - w) = \tau(u) - \tau(w)$.*

Since Assumptions 5.4–5.7 do not rule out the possibility that $\mathrm{N}\tau(u) = \mathrm{N}\tau(w)$ for $u \neq w$, it would seem likely that $B_\mu(u, \tau, \mathrm{N}^\alpha) = B_\mu(w, \tau, \mathrm{N}^\alpha)$ for some $u \neq w$. In order to ferret out the redundant members of $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$, we begin by making the following definition.

Definition 5.14. *For u and w in $\mathbb{Z}^{\mathrm{N}}$, we say that $u \sim w$ if $\mathrm{N}\tau(u) = \mathrm{N}\tau(w)$ and $p \,|\, (u - w)$.*

This equivalence relation is additive in the following sense.

Lemma 5.15. *If u, w, and z are in $\mathbb{Z}^{\mathrm{N}}$ and $u \sim w$, then $u + z \sim w + z$.*

Proof: By hypothesis, p divides $(u + z) - (w + z)$, and by Corollary 5.13, both $\tau(u) - \tau(w)$ and $\tau(u + z) - \tau(w + z)$ equal $\tau(u - w)$, so $\mathrm{N}\tau(u + z) - \mathrm{N}\tau(w + z) = \mathrm{N}\tau(u) - \mathrm{N}\tau(w) = 0$. ■

Lemma 5.16. *If $p \,|\, (u - w)$, then $B_\mu(w, \tau, \mathrm{N}^\alpha) = B_\mu(\cdot + \mathrm{N}\tau(u - w) \,|\, u, \tau, \mathrm{N}^\alpha)$.*

Proof: If $w = u + p(\nu)i_\nu$, then the result follows from Definition 4.1, equation (3.8), and Lemma 5.11, since the knots $\tau_\nu(w), \ldots, \tau_\nu(w + \alpha)$ are obtained from $\tau_\nu(u), \ldots, \tau_\nu(u + \alpha)$ by a shift of $\tau_\nu(p)$. The general result follows from repeated applications of this special case. ■

Lemma 5.16 implies that each spline in $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$ is a shift of one of the finitely many splines

$$\{B_\mu(z, \tau, \mathrm{N}^\alpha) : 0 \leq z < p\}$$

by a point in $L := \{\mathrm{N}\tau(z) : p \,|\, z\}$. By Lemma 5.11, L is the lattice generated by the set of vectors $\{\nu\tau_\nu(p) : \nu \in \mathrm{N}\}$. Take H to be a basis for L (that is, $\#\mathrm{H} = d$ and $L = \mathrm{H}\mathbb{Z}^d$). Then, to each z in $\mathbb{Z}^{\mathrm{N}}$, there corresponds a $u \in \mathbb{Z}^d$ such that

$$B_\mu(z, \tau, \mathrm{N}^\alpha) = B_\mu(\cdot - \mathrm{H}u \,|\, z, \tau, \mathrm{N}^\alpha).$$

That is, after a linear change of variables, S is spanned by finitely many compactly supported functions and their integer shifts.

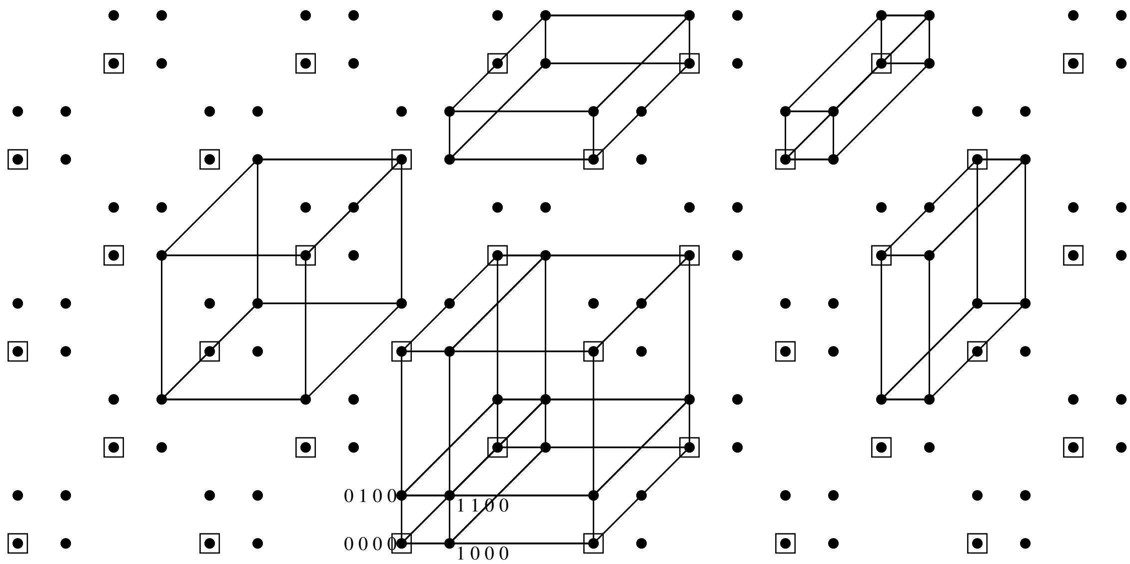

Figure 5.17

Example 5.18. Figure 5.17 shows the knots $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$ (marked $\bullet$) corresponding to the directions

$$\mathrm{N} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and the knot sequences

$$t_{\binom{1}{0}} = t_{\binom{0}{1}} = \{\ldots, -4, -3, 0, 1, 4, 5, 8, 9, \ldots\}$$
$$t_{\binom{1}{1}} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\},$$

for which $p = (2, 2, 1)$. Let $\alpha = (1, 1, 1)$. Figure 5.17 shows the supports of several splines in $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$, including the four splines

$$\{B_\mu(z, \tau, \mathrm{N}^\alpha) : 0 \leq z < p\}. \tag{5.19}$$

The four nonnegative z less than p are printed in the figure beside the corresponding knots $\mathrm{N}\tau(z)$. As noted after Lemma 5.16, any spline in $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$ is a shift of one of the four splines (5.19) by some point in the lattice

$$L = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix} \mathbb{Z}^3 = \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2,$$

marked $\boxdot$ in the figure. After the linear change of variables

$$\begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix} y = x,$$

the functions $B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$ are the $\mathbb{Z}^2$-shifts of the four functions

$$\{E_\mu(z, \tau, \mathrm{N}^\alpha) : 0 \leq z < p\}$$

where $E_\mu(y \mid z, \tau, \mathrm{N}^\alpha) := B_\mu(x \mid z, \tau, \mathrm{N}^\alpha)$.

Theorem 5.20. *Under Assumptions 5.4–5.7, $B_\mu(u,\tau,\mathrm{N}^\alpha) = B_\mu(w,\tau,\mathrm{N}^\alpha)$ if and only if $u \sim w$.*

Proof: If $u \sim w$, then by Corollary 5.13 and Lemma 5.16, $B_\mu(u,\tau,\mathrm{N}^\alpha) = B_\mu(w,\tau,\mathrm{N}^\alpha)$.

To prove the converse, assume $B_\mu(u,\tau,\mathrm{N}^\alpha) = B_\mu(w,\tau,\mathrm{N}^\alpha)$. As a consequence of Definition (4.1) and Corollary 3.10,

$$\mathrm{N}\tau(z) \in \operatorname{supp} B_\mu(z,\tau,\mathrm{N}^\alpha) \subset \mathrm{N}\tau(z) + \operatorname{cone}\mathrm{N}.$$

Because $\operatorname{hull}\mathrm{N}$ is assumed not to contain the zero vector, $B_\mu(u,\tau,\mathrm{N}^\alpha) = B_\mu(w,\tau,\mathrm{N}^\alpha)$ implies $\mathrm{N}\tau(u) = \mathrm{N}\tau(w)$. By Assumption 5.6, there is a $u' \in \mathbb{Z}^\mathrm{N}$ such that $\tau(u) = \tau(u')$ and $p \,|\, (u' - w)$, so that $u' \sim w$. Consequently, by the first half of this theorem, $B_\mu(u',\tau,\mathrm{N}^\alpha) = B_\mu(w,\tau,\mathrm{N}^\alpha) = B_\mu(u,\tau,\mathrm{N}^\alpha)$. By (4.2), the functionals $\delta^\alpha_{u,\mu}$ and $\delta^\alpha_{u',\mu}$ are the same, since they have the same Peano kernel. We'll complete the proof by assuming $u' \neq u$ and producing a function f such that $\delta^\alpha_{u,\mu} f \neq \delta^\alpha_{u',\mu} f$.

Let m be the largest multiinteger for which $\tau(u) = \tau(m)$. By Assumption 5.7, both $m-u$ and $m-u'$ are less than α. Consequently, for f any function, $\delta^\alpha_{u,\mu} f$ is a linear combination of

$$\{D^\gamma_\mathrm{N} f(\mathrm{N}\tau(u)) : 0 \le \gamma \le m-u\}$$

as well as derivative and function values of f at points other than $\mathrm{N}\tau(u)$. Corollary 3.10 implies that the coefficient of $D^{m-u}_\mathrm{N} f(\mathrm{N}\tau(u))$ in this expansion is nonzero. Likewise, $\delta^\alpha_{u',\mu} f$ consists of a linear combination of

$$\{D^\gamma_\mathrm{N} f(\mathrm{N}\tau(u)) : 0 \le \gamma \le m-u'\},$$

not excluding $D^{m-u'}_\mathrm{N} f(\mathrm{N}\tau(u))$, plus terms involving points other than $\mathrm{N}\tau(u)$. Denote by T the points in the supports of $\delta^\alpha_{u,\mu}$ and $\delta^\alpha_{u',\mu}$ other than $\mathrm{N}\tau(u)$.

For any $\beta \in \mathbb{Z}^\mathrm{N}_+$, define the polynomial $(\cdot)^\beta_\mathrm{N}$ by the rule

$$(x)^\beta_\mathrm{N} = \prod_\mathrm{N} (\nu \cdot x)^{\beta(\nu)} \qquad (x \in \mathbb{R}^d).$$

Note $D^\alpha_\mathrm{N} = (D)^\alpha_\mathrm{N}$, so that $D^\alpha_\mathrm{N}(x)^\alpha_\mathrm{N}$ is a positive constant. For each ν in N, the polynomial $(\cdot)^\beta_\mathrm{N}$ has a zero of multiplicity $\beta(\nu)$ along the subspace $\nu^\perp$ of all vectors in $\mathbb{R}^d$ perpendicular to ν. That is, all of its derivatives of total order less than $\beta(\nu)$ vanish on $\nu^\perp$. Because no two elements of N are parallel, $(\cdot)^\beta_\mathrm{N} = (\cdot)^\gamma_\mathrm{N}$ if and only if $\beta = \gamma$.

If $|m-u| > |m-u'|$, let f be a smooth function that agrees with $(\cdot - \mathrm{N}\tau(u))^{m-u}_\mathrm{N}$ in a neighborhood of $\mathrm{N}\tau(u)$ and is identically zero in a neighborhood of T. Then $\delta^\alpha_{u,\mu} f \neq 0 = \delta^\alpha_{u',\mu} f$.

Similarly, if $|m-u| < |m-u'|$, there exists a function at which $\delta^\alpha_{u,\mu}$ and $\delta^\alpha_{u',\mu}$ differ.

Finally, if $|m-u| = |m-u'|$, choose p a linear combination of $(\cdot)^{m-u}_\mathrm{N}$ and $(\cdot)^{m-u'}_\mathrm{N}$ so that $D^{m-u}_\mathrm{N} p = 1$ while $D^{m-u'}_\mathrm{N} p = 0$. Take f to be $p(\cdot - \mathrm{N}\tau(u))$ near $\mathrm{N}\tau(u)$ and 0 near T, and then $\delta^\alpha_{u,\mu} f \neq 0 = \delta^\alpha_{u',\mu} f$.

In any case, we reach a contradiction and the proof is complete. ∎

In light of Theorem 5.20, if $Z \subset \mathbb{Z}^\mathrm{N}$ contains exactly one member of each of the equivalence classes of $\mathbb{Z}^\mathrm{N}$ induced by $\sim$, then the spanning set $B_\mu(Z,\tau,\mathrm{N}^\alpha)$ for S contains no duplicate members. In the final result of this section, we construct such a set Z.

Theorem 5.21. *Under Assumptions 5.4–5.7, If H is any basis in N, and if $\mathrm{K} := \mathrm{N} \setminus \mathrm{H}$, then for every κ in K, there exists a positive integer n_κ such that the set*

$$\mathbb{Z}^\mathrm{H} \times \{0\,.\,.\,n - \mathbb{1}\}^\mathrm{K} := \{z \in \mathbb{Z}^\mathrm{N} : z(\kappa) \in \{0\,.\,.\,n_\kappa - 1\} \forall \kappa \in \mathrm{K}\}$$

contains exactly one member of each of the $\sim$-equivalence classes of $\mathbb{Z}^\mathrm{N}$.

For instance, suppose that each sequence t_ν is increasing, so that $u \sim w$ iff $\mathrm{N}\tau(u) = \mathrm{N}\tau(w)$. Theorem 5.21 then implies that each knot $\mathrm{N}\tau(z)$ can be written uniquely as the sum of a point in the d-dimensional mesh $\mathrm{H}\tau(\mathbb{Z}^\mathrm{H})$ and one of the finitely many points in $\mathrm{K}\tau(\{0\,.\,.\,n - \mathbb{1}\}^\mathrm{K})$.

Example 5.22. Let $N = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ and let τ consist of the sequences

$$t_{\binom{1}{0}} = t_{\binom{0}{1}} = \{\ldots, -4, -3, 0, 1, 4, 5, \ldots\}$$
$$t_{\binom{1}{1}} = \{\ldots, -4, 0, 4, 8, \ldots\}$$
$$t_{\binom{-1}{1}} = \{\ldots, -4, -3.5, 0, .5, 4, 4.5, \ldots\}$$

Figures 5.23-5.26 show the corresponding knots $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$ and $\mathrm{H}\tau(\mathbb{Z}^{\mathrm{H}})$ for various $\mathrm{H} \in \mathcal{B}_d(\mathrm{N})$. The knots $\mathrm{H}\tau(\mathbb{Z}^{\mathrm{H}})$ appear as filled circles connected by solid gridlines in the directions of H. When a copy of the grid $\mathrm{K}\tau(\{0\,..\,n_\kappa - 1\}^{\mathrm{K}})$, drawn with dashed lines, is anchored at each point of $\mathrm{H}\tau(\mathbb{Z}^{\mathrm{H}})$, the entire knot set $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$ is generated without overlap.

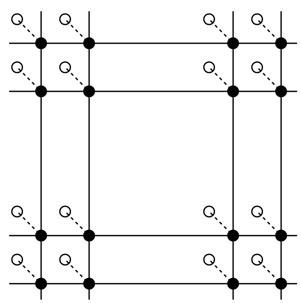

Figure 5.23: $\mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

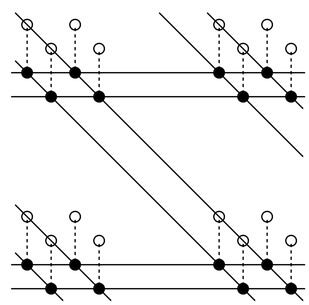

Figure 5.24. $\mathrm{H} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

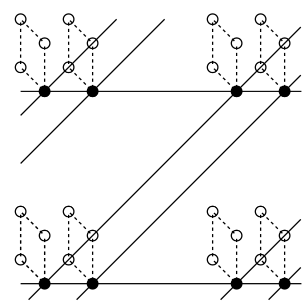

Figure 5.25. $\mathrm{H} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Figures 5.23-5.25 illustrate that

$$\text{when H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad n_{\binom{1}{1}} = 1, \quad n_{\binom{-1}{1}} = 2;$$
$$\text{when H} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad n_{\binom{1}{1}} = 1, \quad n_{\binom{0}{1}} = 2; \text{ and}$$
$$\text{when H} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad n_{\binom{0}{1}} = 2, \quad n_{\binom{-1}{1}} = 2;$$

The choice of $\{n_\kappa\}_{\mathrm{K}}$ is not unique, as Figure 5.26 illustrates. When $\mathrm{H} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, one can choose $n_{\binom{1}{0}} = 4$ and $n_{\binom{0}{1}} = 2$ or vice versa (depending on how one orders the elements of $\mathrm{N} \setminus \mathrm{H}$ in the inductive proof of Theorem 5.21).

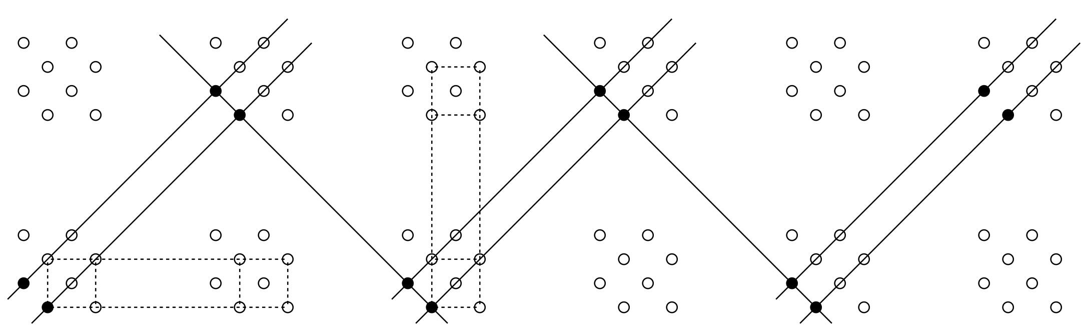

Figure 5.26: $\mathrm{H} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

The proof of Theorem 5.21 relies on the following lemma.

Lemma 5.27. *Assume the same hypotheses as Theorem 5.21. For* H *any basis in* N *and* κ *any element in* $\mathrm{N} \setminus \mathrm{H}$, *there exists a natural number* n *such that* $p(\kappa) \mid n$ *and* $t_\kappa(n)\kappa \in \mathrm{H}\tau(\mathbb{Z}^{\mathrm{H}})$

Proof of Lemma 5.27: Let $\widehat{\mathrm{H}} := \{\tau_\eta(p)\eta : \eta \in \mathrm{H}\}$ and $\hat{\kappa} := \tau_\kappa(p)\kappa$. Since $\widehat{\mathrm{H}}$ is an invertible rational matrix, there exists $s \in \mathbb{Q}^{\mathrm{H}}$ such that $\hat{\kappa} = \widehat{\mathrm{H}}s$. Choose a natural number m such that $ms \in \mathbb{Z}^{\mathrm{H}}$. Then

$$m\tau_\kappa(p)\kappa = \sum_{\mathrm{H}} ms(\eta)\tau_\eta(p)\eta.$$

By Lemma 5.11, this can be rewritten

$$\tau_\kappa(mp)\kappa = \sum_{\mathrm{H}} \tau_\eta(ms(\eta)p)\eta.$$

Taking $n = mp(\kappa)$ completes the proof. ■

Proof of Theorem 5.21: The construction is inductive. Assume $\mathrm{K}' \subset \mathrm{K}$ and $\forall \kappa \in \mathrm{K}'$, $\exists n_\kappa \in \mathbb{N}$ such that

$$\mathbb{Z}^{\mathrm{H}} \times \{0 \mathinner{.\,.} n - \mathbb{1}\}^{\mathrm{K}'}$$

contains at least one member of each $\sim$-class of $\mathbb{Z}^{\mathrm{H}\cup\mathrm{K}'}$. (This is trivially true in the start-up case that $\mathrm{K}' = \emptyset$.)

Fix $\kappa \in \mathrm{K} \setminus \mathrm{K}'$. Pick n_κ to be the smallest natural number such that $p(\kappa) \mid n_\kappa$ and

$$t_\kappa(n_\kappa)\kappa \in (\mathrm{H} \cup \mathrm{K}')\tau(\mathbb{Z}^{\mathrm{H}\cup\mathrm{K}'}).$$

The existence of such an n_κ is guaranteed by Lemma 5.27. Let $l \in \mathbb{Z}^{\mathrm{N}}$ be supported on $\mathrm{H} \cup \mathrm{K}'$ and satisfy $(\mathrm{H} \cup \mathrm{K}')\tau(l) = t_\kappa(n_\kappa)\kappa$. By Assumption 5.6, it can also be assumed that $p(\eta) \mid l(\eta)$ for every $\eta \in \mathrm{H} \cup \mathrm{K}'$.

We claim that each z in $\mathbb{Z}^{\mathrm{H}\cup\mathrm{K}'\cup\kappa}$ is equivalent to some member of $\mathbb{Z}^{\mathrm{H}} \times \{0 \mathinner{.\,.} n - \mathbb{1}\}^{\mathrm{K}'\cup\kappa}$. To see this, choose $m \in \mathbb{Z}$ such that $z(\kappa) + mn_\kappa \in \{0 \mathinner{.\,.} n_\kappa - 1\}$. Define $u := z + mn_\kappa i_\kappa - ml \in \mathbb{Z}^{\mathrm{H}\cup\mathrm{K}'} \times \{0 \mathinner{.\,.} n_\kappa - 1\}$. Corollary 5.13 and Lemma 5.15 then imply $u \sim z$. By the inductive hypothesis, there exists $v \in \mathbb{Z}^{\mathrm{H}} \times \{0 \mathinner{.\,.} n - \mathbb{1}\}^{\mathrm{K}'}$ such that $v \sim u_{\mathrm{H}\cup\mathrm{K}'}$. Extend v to $\mathrm{H} \cup \mathrm{K}' \cup \kappa$ by defining $v(\kappa) = u(\kappa)$. Then $v \sim u \sim z$, proving the claim.

By induction, the assignment of an n_κ to each $\kappa \in \mathrm{K}$ is complete, and every z in $\mathbb{Z}^{\mathrm{N}}$ is equivalent to some member of $\mathbb{Z}^{\mathrm{H}} \times \{0 \mathinner{.\,.} n - \mathbb{1}\}^{\mathrm{K}}$. It remains to be shown that if u and w are in this set and $u \sim w$, then $u = w$.

Assume $u \neq w$. Order the elements of N by first listing the members of H (in any order) and then the members of K in the order used in the construction above. Let ν be the last element of N for which $u(\nu) \neq w(\nu)$.

If $\nu \in \mathrm{H}$, then $u = w$ on K and therefore $\mathrm{H}\tau(u_{\mathrm{H}}) = \mathrm{H}\tau(w_{\mathrm{H}})$. Since H is a basis, $\tau(u_{\mathrm{H}}) = \tau(w_{\mathrm{H}})$. By assumption, $u \sim w$, so there exists a nonzero integer m for which $u(\nu) = w(\nu) + mp(\nu)$. Lemma 5.11 and Corollary 5.13 imply $\tau_\nu(u) = \tau_\nu(w) + m\tau_\nu(p)$, so that $\tau_\nu(p) = 0$, contradicting Corollary 5.12.

Otherwise, $\nu \in \mathrm{K}$. Let K' consist of those points in K preceding ν in the ordering of N. Without loss of generality, $0 < w(\nu) - u(\nu) < n_\nu$. Since $p \mid (w - u)$, Corollary 5.13 implies

$$\tau_\nu(w - u)\nu \in (\mathrm{H} \cup \mathrm{K}')\tau(\mathbb{Z}^{\mathrm{H}\cup\mathrm{K}'}),$$

contradicting our choice of n_ν.

Thus, no two elements of $\mathbb{Z}^{\mathrm{H}} \times \{0 \mathinner{.\,.} n - \mathbb{1}\}^{\mathrm{K}}$ are equivalent, and Theorem 5.21 is proven. ■

As we shall see in the next section (Lemma 6.16), one consequence of Theorem 5.21 is that, not only are the splines of $B_\mu(Z, \tau, \mathrm{N}^\alpha)$ distinct, but only finitely many are nonzero at any x in $\mathbb{R}^d$, so that every series of the form

$$\sum_Z a_z B_\mu(z, \tau, \mathrm{N}^\alpha)$$

converges pointwise.

6. The span and linear independence of $B_\mu(Z, \tau, \mathrm{N}^\alpha)$

By **exponential polynomial**, we shall mean a sum of functions of the form $e_\theta p$ for $\theta \in \mathbb{R}^d$ and p a polynomial.

Theorem 6.1. *Under Assumptions 5.4–5.7, the exponential polynomials in S ($= \operatorname{span} B_\mu(\mathbb{Z}^{\mathrm{N}}, \tau, \mathrm{N}^\alpha)$) are the space $D_\mu(\mathrm{N}^\alpha)$.*

Proof: Since each element of S is piecewise in $D_\mu(\mathrm{N}^\alpha)$, it follows that any exponential-polynomial in S must belong to this space. It therefore will suffice to show $D_\mu(\mathrm{N}^\alpha) \subset S$.

Let $\widehat{\mathrm{N}} := \{\hat\nu := \tau_\nu(p)\nu : \nu \in \mathrm{N}\}$, so that the lattice $\widehat{\mathrm{N}}\mathbb{Z}^{\mathrm{N}} \subseteq \mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})$. Pick q a natural number so that $q\mathbb{Z}^d \subseteq \widehat{\mathrm{N}}\mathbb{Z}^{\mathrm{N}}$ and $\tau_\nu(p) \,|\, q$ for every $\nu \in \mathrm{N}$.

Since $\tau(z) := qz$ is a subsequence of $\tau(z)$, the exponential box spline $C_{q\mu}(q\mathrm{N}^\alpha)$ lies in S, and, by Lemma 5.16, so does its shifts by points in $\widehat{\mathrm{N}}\mathbb{Z}^{\mathrm{N}}$. Because this lattice contains $q\mathbb{Z}^d$,

$$\begin{aligned} S \supset\ & \operatorname*{span}_{z\in\mathbb{Z}^d} C_{q\mu}(\cdot - qz \,|\, q\mathrm{N}^\alpha) \\ =\ & \operatorname*{span}_{z\in\mathbb{Z}^d} C_{q\mu}(q^{-1}\cdot - z \,|\, \mathrm{N}^\alpha). \end{aligned}$$

As is well known [1,9], the integer translates of $C_{q\mu}(\mathrm{N}^\alpha)$ span $D_{q\mu}(\mathrm{N}^\alpha)$, and so the proof is complete. ∎

Let H be a basis in N, define $\mathrm{K} := \mathrm{N} \setminus \mathrm{H}$, and pick $\theta \in \mathbb{R}^d$ so that $e_\theta \in D_\mu(\mathrm{H}^{\alpha_{\mathrm{H}}})$. By Theorem 6.1,

$$e_\theta \in \operatorname{span} B_{\mu_{\mathrm{H}}}(\mathbb{Z}^{\mathrm{H}}, \tau, \mathrm{H}^{\alpha_{\mathrm{H}}}). \tag{6.2}$$

We will not distinguish between $u \in \mathbb{Z}^{\mathrm{H}}$ and its extension to N by the rule $u(\nu) = 0$ if $\nu \notin \mathrm{H}$. Let v be in $\{0\,..\,n - \mathbb{1}\}^{\mathrm{K}}$, and similarly extend v to all of N. From Definition (4.1), it follows that

$$B_\mu(u + v, \tau, \mathrm{N}^\alpha) = B_{\mu_{\mathrm{H}}}(u, \tau, \mathrm{H}^{\alpha_{\mathrm{H}}}) * B_{\mu_{\mathrm{K}}}(v, \tau, \mathrm{K}^{\alpha_{\mathrm{K}}}) \tag{6.3}$$

where, as in Definition 4.1, $B_{\mu_{\mathrm{K}}}(v, \tau, \mathrm{K}^{\alpha_{\mathrm{K}}})$ is the convolution of the distributions

$$\left\{\phi \mapsto \int_{-\infty}^{\infty} \phi(x\kappa) B_{\mu_\kappa}(x \,|\, \tau_\kappa(v), \ldots, \tau_\kappa(v + \alpha))\, dx \ : \ \kappa \in \mathrm{K}\right\}. \tag{6.4}$$

By equation (3.14), the convolution of e_θ with any of the distributions (6.4) is a nonzero multiple of e_θ. Hence (6.2) and (6.3) imply that

$$\forall v \in \{0\,..\,n - \mathbb{1}\}^{\mathrm{K}}, \quad e_\theta \in \operatorname{span} B_\mu(\mathbb{Z}^{\mathrm{H}} + v, \tau, \mathrm{N}^\alpha) \tag{6.5}$$

and we have proved the following result.

Lemma 6.6. *Under the same hypotheses as Theorem 6.1, if there is a basis* H *in* N *and* $\kappa \in \mathrm{N} \setminus \mathrm{H}$ *for which* $n_\kappa > 1$, *then* $B_\mu(Z, \tau, \mathrm{N}^\alpha)$ *is linearly dependent.*

Since $p(\kappa) \,|\, n_\kappa$ for each $\kappa \in \mathrm{N} \setminus \mathrm{H}$, Lemma 6.6 has the following corollary.

Corollary 6.7. *If* t_ν *is a nonuniform sequence (i.e., if* $p(\nu) > 1$*) for some* ν *in* N *and if* $B_\mu(Z, \tau, \mathrm{N}^\alpha)$ *is linearly independent, then every basis* H *in* N *contains* ν.

In other words, in order for $B_\mu(Z, \tau, \mathrm{N}^\alpha)$ to be linearly independent, its direction set N must equal $\mathrm{N}_1 \cup \mathrm{N}_2$ where $\operatorname{span} \mathrm{N}_1 \cap \operatorname{span} \mathrm{N}_2 = \{0\}$, the elements of N_2 are linearly independent, and t_ν is uniform for each ν in N_1.

Recall that a set of functions $\mathcal{F}$ is said to be locally linearly independent provided that, for any open set Ω in $\mathbb{R}^d$, if $\sum_{\mathcal{F}} a_f f$ is identically zero on Ω, then $a_f = 0$ for each f whose support intersects Ω. Clearly $\mathcal{F}$ is linearly independent if it is locally so. Therefore, we have so far proven the following.

$$\begin{aligned} & B_\mu(Z, \tau, \mathrm{N}^\alpha) \text{ is locally linearly independent} \\ \Longrightarrow\ & B_\mu(Z, \tau, \mathrm{N}^\alpha) \text{ is linearly independent} \\ \Longrightarrow\ & \forall \mathrm{H} \in \mathcal{B}_d(\mathrm{N}),\ \forall \kappa \in \mathrm{N} \setminus \mathrm{H},\ n_\kappa = 1. \end{aligned} \tag{6.8}$$

The remainder of this section will be devoted to showing that these three conditions are equivalent. Toward that end, we recall a result from the box spline literature [4,8,28] and its proof. For brevity, we adopt the convention [4] of letting $\square$ denote the unit cube $[0\,..\,1]^S$ with the set S to be made clear by context. For instance, $A\square$ stands for $A[0\,..\,1]^A$ for any matrix A.

Theorem 6.9. *Let the columns of* N *span* $\mathbb{R}^d$. *Then to every basis* H *in* N *there is associated a vector* $a_\mathrm{H} \in \{0,1\}^\mathrm{N}$ *such that* $\mathrm{N}\square$ *is the essentially disjoint union of the sets*

$$\mathrm{N}a_\mathrm{H} + \mathrm{H}\square \qquad (\mathrm{H} \in \mathcal{B}_d(\mathrm{N})).$$

That is, the support of the box spline $C_0(\mathrm{N})$ can be partitioned into exactly $\#\mathcal{B}_d(\mathrm{N})$ parallelograms generated by the bases in N. See Figure 6.10, for example.

Note that in Theorem 6.9, N may be a multiset. In particular, Theorem 6.9 applies to N^α if N satisfies Assumption 5.4 and $\alpha \in \mathbb{Z}_+^\mathrm{N}$

Sketch of proof: The proof is by induction on $\#\mathrm{N} \geq d$, the startup case being trivial. For the inductive step, it is observed that

$$(\mathrm{N} \cup \zeta)\square \setminus \mathrm{N}\square = G + \zeta[0\,.\,.\,1]$$

where $\zeta[0\,.\,.\,1]$ is the closed line segment from 0 to ζ and where G is that part of the boundary of $\mathrm{N}\square$ at which ζ is outward pointing. It is proven that G is the essentially disjoint union of sets of the form

$$(\mathrm{N} \cap \mathcal{H})\square + \sum_{\xi \in \mathrm{N}\setminus\mathcal{H}} g_\mathcal{H}(\xi)\xi \qquad (\mathcal{H} \in \mathbb{H}(\mathrm{N}))$$

where $g_\mathcal{H}(\xi)$ is either 0 or 1, depending on whether of not ξ and ζ lie on opposite sides of the hyperplane $\mathcal{H}$. Therefore $G + \zeta[0\,.\,.\,1]$ can be partitioned by applying the inductive hypothesis to each of the sets

$$\big((\mathrm{N} \cap \mathcal{H}) \cup \zeta\big)\square + (N \setminus \mathcal{H})g_\mathcal{H} \qquad (\mathcal{H} \in \mathbb{H}(\mathrm{N})). \qquad \blacksquare$$

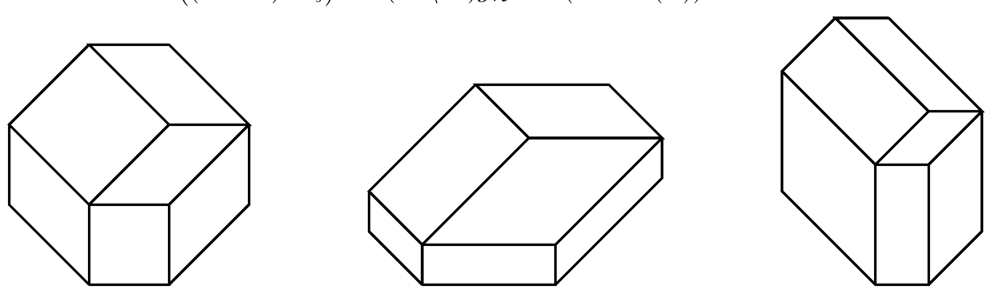

Figure 6.10: The partitions of Thm. 6.9 with $\mathrm{N} = \begin{pmatrix} a & 0 & c & -d \\ 0 & b & c & d \end{pmatrix}$ **for various** $a, b, c, d > 0$**.**

Upon inspection of this proof, we observe the following corollary.

Corollary 6.11. *The vectors* a_H *in Theorem 6.9 can be chosen so that* $a_\mathrm{H}(\eta) = 0$ *for all* $\eta \in \mathrm{H}$. *Furthermore, let* $\{\phi(\nu) : \nu \in \mathrm{N}\}$ *be a collection of positive numbers, and define for all* $\mathrm{M} \subseteq \mathrm{N}$

$$\Phi(\mathrm{M}) := \{\phi(\mu)\mu : \mu \in \mathrm{M}\}.$$

Then $\Phi(\mathrm{N})\square$ *is the essentially disjoint union of the sets*

$$\Phi(\mathrm{N})a_\mathrm{H} + \Phi(\mathrm{H})\square \qquad (\mathrm{H} \in \mathcal{B}_d(\mathrm{N})).$$

That is, a_H is invariant under rescalings of N. See Figure 6.10.

Proof: That a_H is supported on $\mathrm{N} \setminus \mathrm{H}$ is trivial in case $\#\mathrm{N} = d$, since then $\mathcal{B}_d(\mathrm{N}) = \{\mathrm{N}\}$ and $a_\mathrm{N} = 0$. Assuming it is true for all matrices with the same number of elements as N, proceed with the inductive step of the construction.

Suppose $\mathrm{H} \in \mathcal{B}_d(\mathrm{N} \cup \zeta)$. If $\mathrm{H} \in \mathcal{B}_d(\mathrm{N})$, then the inductive hypothesis already implies $a_\mathrm{H} = 0$ on H. On the other hand, if $\mathrm{H} \in \mathcal{B}_d(\mathrm{N} \cup \zeta) \setminus \mathcal{B}_d(\mathrm{N})$, then there exists a unique hyperplane $\mathcal{H}$ in $\mathbb{H}(\mathrm{N})$ such that H is in $\big((\mathrm{N} \cap \mathcal{H}) \cup \zeta\big)$. Hence, in the partition of

$$\big((\mathrm{N} \cap \mathcal{H}) \cup \zeta\big)\square + (\mathrm{N} \setminus \mathcal{H})g_\mathcal{H}$$

the parallelogram corresponding to H has the form

$$\big((\mathrm{N}\cap\mathcal{H})\cup\zeta\big)a_{\mathrm{H}}+(\mathrm{N}\setminus\mathcal{H})g_{\mathcal{H}}+\mathrm{H}\square.$$

By definition, $g_{\mathcal{H}}$ is supported on $\mathrm{N}\setminus\mathcal{H}\subset\mathrm{N}\setminus\mathrm{H}$, while a_{H} is supported on $\big((\mathrm{N}\cap\mathcal{H})\cup\zeta\big)\setminus\mathrm{H}$ by the induction hypothesis. Therefore, the proof of the corollary's first part is complete.

The second part of this corollary is again trivial in case $\#\mathrm{N}=d$. In the inductive step, observe that the hyperplanes of N and of $\Phi(\mathrm{N})$ are the same. Furthermore, $g_{\mathcal{H}}(\xi)=g_{\mathcal{H}}(\Phi(\xi))$ for all $\mathcal{H}$ in $\mathbb{H}(\mathrm{N})$ and $\xi\in\mathrm{N}$. Applying the inductive hypothesis to the partion of each of the sets

$$\Phi\big((\mathrm{N}\cap\mathcal{H})\cup\zeta\big)\square$$

completes the proof. ■

Corollary 6.12. *Assume the same hypotheses as Definition 4.1. Assume also that* $\alpha>0$. *Then for each basis* H *in* N^α, *there exists a multiindex* $c_{\mathrm{H}}\in\mathbb{Z}_+^{\mathrm{N}}$ *such that, for any* $z\in\mathbb{Z}^{\mathrm{N}}$, *the set*

$$\mathrm{N}[\tau(z)\,..\,\tau(z+\alpha)]$$

is the essentially disjoint union of the parallelograms

$$\mathrm{N}[\tau(z+c_{\mathrm{H}})\,..\,\tau(z+c_{\mathrm{H}}+\mathbb{1}_{\mathrm{H}})]\qquad(\mathrm{H}\in\mathcal{B}_d(\mathrm{N}^\alpha)).$$

Just as Theorem 6.9 partitions the support of a box spline, Corollary 6.12 partitions the support of $B_\mu(z,\tau,\mathrm{N}^\alpha)$ into parallelograms, one for each member of $\mathcal{B}_d(\mathrm{N}^\alpha)$. (Due to multiple knots, some of these parallelograms may have zero volume.) Furthermore, each of these parallelograms is anchored at the point $\mathrm{N}\tau(z+c_{\mathrm{H}})$—the c_{H}th knot after $\mathrm{N}\tau(z)$—with c_{H} independent of z. For example, see Figure 6.14, which is based on the N and τ in Example 5.22.

Proof of Corollary 6.12: For each basis H in N, choose a_{H} as in Theorem 6.9, with a_{H} supported on $\mathrm{N}\setminus\mathrm{H}$. For every multiindex β supported on H with $\beta<\alpha$ and for every $\nu\in\mathrm{N}$, define

$$b_{\mathrm{H}}^{\beta}(\nu):=\alpha(\nu)a_{\mathrm{H}}(\nu)+\beta(\nu).$$

Since $a_{\mathrm{H}}(\nu)$ is 0 or 1, $b_{\mathrm{H}}^{\beta}\le\alpha$, with strict inequality on H. Fix z in $\mathbb{Z}^{\mathrm{N}}$ and define $\phi(\nu)=\tau_\nu(z+\alpha)-\tau_\nu(z)$ for all ν in N. Then

$$\mathrm{N}[\tau(z)\,..\,\tau(z+\alpha)]=\mathrm{N}\tau(z)+\Phi(\mathrm{N})\square.$$

By Corollary 6.11, this is the essentially disjoint union of parallelograms

$$\bigcup_{\mathrm{H}\in\mathcal{B}_d(\mathrm{N})}(\mathrm{N}\tau(z)+\Phi(\mathrm{N})a_{\mathrm{H}}+\Phi(\mathrm{H})\square).\tag{6.13}$$

For any ν in N,

$$\tau_\nu(z)+(\tau_\nu(z+\alpha)-\tau_\nu(z))a_{\mathrm{H}}(\nu)=\tau_\nu(z+b_{\mathrm{H}}^{0})$$

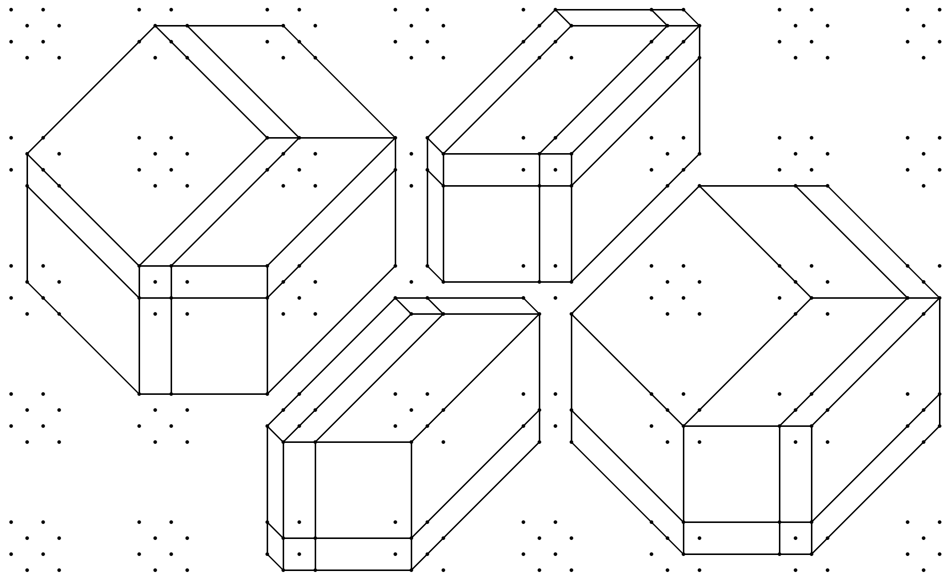

Figure 6.14: Corollary 6.12 with $\mathrm{N} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ **and** $\alpha = (2, 2, 1, 1)$**.**

and therefore (6.13) is

(6.15) $$\bigcup_{\mathrm{H} \in \mathcal{B}_d(\mathrm{N})} \big(\mathrm{N}\tau(z + b^0_{\mathrm{H}}) + \mathrm{H}[0 \,..\, \tau(z + \alpha_{\mathrm{H}}) - \tau(z)]\big).$$

Because α_{H} is supported on H and b^0_{H} is supported on $\mathrm{N} \setminus \mathrm{H}$,

$$\mathrm{H}\tau(z + \alpha_{\mathrm{H}}) - \mathrm{H}\tau(z) = \mathrm{N}\tau(z + \alpha_{\mathrm{H}}) - \mathrm{N}\tau(z) = \mathrm{N}\tau(z + b^0_{\mathrm{H}} + \alpha_{\mathrm{H}}) - \mathrm{N}\tau(z + b^0_{\mathrm{H}}).$$

Therefore (6.15) equals

$$\bigcup_{\mathrm{H} \in \mathcal{B}_d(\mathrm{N})} \mathrm{N}[\tau(z + b^0_{\mathrm{H}}) \,..\, \tau(z + b^0_{\mathrm{H}} + \alpha_{\mathrm{H}})],$$

a union of parallelograms with sides parallel to the elements of H. To complete the proof, divide each of these into the essentially disjoint parts

$$\bigcup_{\mathrm{H} \in \mathcal{B}_d(\mathrm{N})} \bigcup_{\beta < \alpha} \mathrm{N}[\tau(z + b^\beta_{\mathrm{H}}) \,..\, \tau(z + b^\beta_{\mathrm{H}} + 1\!\!1_{\mathrm{H}})]$$

and observe that each H in $\mathcal{B}_d(\mathrm{N})$ appears $\#\{\beta \in \mathbb{Z}^{\mathrm{H}}_+ : \beta < \alpha_{\mathrm{H}}\}$ times in $\mathcal{B}_d(\mathrm{N}^\alpha)$. ■

Lemma 6.16. *Under the same hypotheses as Theorem 5.21, if x is in $\mathbb{R}^d \setminus \Gamma(\mathrm{N}, \tau)$, then the number of splines in $B_\mu(Z, \tau, \mathrm{N}^\alpha)$ with support at x is*

$$\sum_{\mathrm{H} \in \mathcal{B}_d(\mathrm{N}^\alpha)} \prod_{\kappa \in \mathrm{N} \setminus \mathrm{H}} n_\kappa.$$

Proof: The number of splines not zero at x is the value at x of the function

$$\sum_{z \in Z} \chi(\operatorname{supp} B_\mu(z, \tau, \mathrm{N}^\alpha)) = \sum_{z \in Z} \chi(\mathrm{N}[\tau(z) \,..\, \tau(z + \alpha)])$$

where Z is any subset of $\mathbb{Z}^{\mathrm{N}}$ containing exactly one member of each of the $\sim$-equivalence classes of $\mathbb{Z}^{\mathrm{N}}$. By Corollary 6.12, this sum is almost everywhere equal to

$$\sum_{z\in Z}\sum_{\mathrm{H}\in\mathcal{B}_d(\mathrm{N}^\alpha)}\chi(\mathrm{N}[\tau(z+c_{\mathrm{H}})\,..\,\tau(z+c_{\mathrm{H}}+\mathbb{1}_{\mathrm{H}})]).$$

Interchange the order of summation and apply Theorem 5.21 to transform this into

$$\sum_{\mathrm{H}\in\mathcal{B}_d(\mathrm{N}^\alpha)}\sum_{v\in\{0..n-\mathbb{1}\}^{\mathrm{N}\setminus\mathrm{H}}}\sum_{u\in\mathbb{Z}^{\mathrm{H}}}\chi(\mathrm{N}[\tau(u+v+c_{\mathrm{H}})\,..\,\tau(u+v+c_{\mathrm{H}}+\mathbb{1}_{\mathrm{H}})]). \tag{6.17}$$

Since, for any H and any ν, the parallelograms

$$\mathrm{N}[\tau(u+v+c_{\mathrm{H}})\,..\,\tau(u+v+c_{\mathrm{H}}+\mathbb{1}_{\mathrm{H}})] \qquad (u\in\mathbb{Z}^{\mathrm{H}})$$

form a partition of $\mathbb{R}^d$, overlapping only on $\Gamma(\mathrm{H},\tau)\subseteq\Gamma(\mathrm{N},\tau)$, the inner sum of (6.17) is equal to 1 on $\mathbb{R}^d\setminus\Gamma(\mathrm{N},\tau)$, completing the proof. ■

Let Ω be an open connected component of $\mathbb{R}^d\setminus\Gamma(\mathrm{N},\tau)$. By Theorem 6.1, the exponential-polynomials

$$\{B_\mu(z,\tau,\mathrm{N}^\alpha)|_\Omega \;:\; z\in Z,\ \mathrm{N}[\tau(z)\,..\,\tau(z+\alpha)]\cap\Omega\neq\emptyset\} \tag{6.18}$$

span $D_\mu(\mathrm{N}^\alpha)$, a space of dimension $\#\mathcal{B}_d(\mathrm{N}^\alpha)$. Lemma 6.16 therefore implies that these exponential-polynomials are linearly independent if $n_\kappa = 1$ for every $\mathrm{H}\in\mathcal{B}_d(\mathrm{N})$ and every $\kappa\in\mathrm{N}\setminus\mathrm{H}$. The linear independence of (6.18) for all such Ω implies the local linear independence of $B_\mu(Z,\tau,\mathrm{N}^\alpha)$, and therefore we have proven the equivalence of the three conditions in (6.8) [cf. 4, II.57].

Theorem 6.19. *Under the same hypotheses as Theorem 5.21, the following conditions are equivalent.*

a. *The splines $B_\mu(Z,\tau,\mathrm{N}^\alpha)$ are linearly independent.*

b. *The splines $B_\mu(Z,\tau,\mathrm{N}^\alpha)$ are locally linearly independent.*

c. *At any $x\in\mathbb{R}^d\setminus\Gamma(\mathrm{N},\tau)$, the number of nonzero splines in $B_\mu(Z,\tau,\mathrm{N}^\alpha)$ is $\#\mathcal{B}_d(\mathrm{N}^\alpha)$.*

d. *For each basis H in N, and for each κ in $\mathrm{N}\setminus\mathrm{H}$, $n_\kappa=1$; that is, $\mathrm{N}\tau(\mathbb{Z}^{\mathrm{N}})=\mathrm{H}\tau(\mathbb{Z}^{\mathrm{H}})$.*

e. *Every H in $\mathcal{B}_d(\mathrm{N})$ contains all those directions ν for which $p(\nu)>1$, and the absolute value of the determinant of $\{\tau_\nu(p)\nu:\nu\in\mathrm{H}\}$ is the same for all H in $\mathcal{B}_d(\mathrm{N})$.*

To prove the equivalence of condition e to the others in Theorem 6.19, note that conditions a-d are equivalent to

$$\forall\mathrm{H}\in\mathcal{B}_d(\mathrm{N}),\quad \forall\nu\in\mathrm{N}\setminus\mathrm{H},\quad p(\nu)=1 \tag{6.20}$$

and

$$\forall\mathrm{H}\in\mathcal{B}_d(\mathrm{N}),\quad \forall\nu\in\mathrm{N}\setminus\mathrm{H},\quad \exists u\in\mathbb{Z}^{\mathrm{H}} \text{ such that } t_\nu(1)\nu=\sum_{\mathrm{H}}\tau_\eta(u)\eta \tag{6.21}$$

By Assumption 5.6, one can assume that $p(\eta)\,|\,u(\eta)$ for every η in H, so Lemma 5.11 implies that (6.21) is equivalent to

$$\forall\mathrm{H}\in\mathcal{B}_d(\mathrm{N}),\quad \forall\nu\in\mathrm{N}\setminus\mathrm{H},\quad \exists z\in\mathbb{Z}^{\mathrm{H}} \text{ such that } t_\nu(1)\nu=\sum_{\mathrm{H}}z(\eta)\tau_\eta(p)\eta.$$

Take a natural number q such that, $q\tau_\nu(p)\nu$ is in $\mathbb{Z}^d$ for each ν in N. Then the above states that the lattice generated by the integer vectors $\{q\tau_\nu(p)\nu:\nu\in\mathrm{N}\}$ equals the lattice generated by $\{q\tau_\eta(p)\eta:\eta\in\mathrm{H}\}$ for any basis H in N, which happens exactly when the absolute value of the determinant of $\{q\tau_\eta(p)\eta:\eta\in\mathrm{H}\}$ is the same for every H in $\mathcal{B}_d(\mathrm{N})$.

As a final note, we point out that, in a special case, Theorem 6.19 reduces to known results.

If $N \subset \mathbb{Z}^d$ and if $\tau(z) = z$ for all z in $\mathbb{Z}^{\mathrm{N}}$, then as was seen in the proof of Theorem 6.1, $B_\mu(z, \tau, \mathrm{N}^\alpha)$ is a constant multiple of $C_\mu(\cdot - \mathrm{N}z \,|\, \mathrm{N}^\alpha)$. If, in addition, $\mathrm{N}\mathbb{Z}^{\mathrm{N}} = \mathbb{Z}^d$, our space S is the usual box spline space spanned by the set (5.1). In this case, condition d is equivalent to

$$\forall \mathrm{H} \in \mathcal{B}_d(\mathrm{N}), \qquad \mathrm{H}\mathbb{Z}^d = \mathbb{Z}^d,$$

the well-known unimodularity condition for the linear independence of (5.1) [9,25].

(If, instead, $\mathrm{N}\mathbb{Z}^{\mathrm{N}}$ is a proper sublattice of $\mathbb{Z}^d$, then the splines (5.1) must be linearly dependent.)

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