Favard's Interpolation Problem in One or More Variables

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Abstract

Given scattered data on the real line, Favard [4] constructed an interpolant which depends linearly and locally on the data and whose nth derivative is locally bounded by the nth divided differences of the data times a constant depending only on n. It is shown that the $(n-1)$th derivative of Favard's interpolant can be likewise bounded by divided differences, and that one can bound at best two consecutive derivatives of any interpolant by the corresponding divided differences. In this sense, Favard's univariate interpolant is the best possible. Favard's result has been extended [8] to a special case in several variables, and here the extent to which this can be repeated in a more general setting is proven exactly.

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1. Introduction

This paper is a discussion of interpolation to function values on discrete sets of data points, and the extent to which the derivatives of an interpolant can be bounded above in absolute value by a constant multiple of its necessary size, as determined by the divided differences of the data. As it turns out, a construction of Favard [4] gives a complete answer to this question in the univariate case. Specifically, let $\mathbf{m} = \{m_i\}_{-\infty}^{\infty}$ be a biinfinite, strictly increasing sequence of real numbers, let f be a function defined on $\mathbf{m}$, and let $[m_s, \ldots, m_{s+n}]f$ denote the nth degree divided difference of f formed from its values on $m_s, \ldots, m_{s+n}$. Favard showed that, for any $\mathbf{m}$, f, and nonnegative integer n, there exists a smooth function F defined on $(\inf \mathbf{m}, \sup \mathbf{m})$, agreeing with f on $\mathbf{m}$, depending locally and linearly on f, for which $|D^n F|$ is locally no larger than the numbers

$$\{|[m_s, \ldots, m_{s+n}]f| : s \in \mathbb{Z}\}$$

times a constant depending only on n.

In several papers [1,2,3], de Boor has estimated this constant as well as the related constant

$$\sup_{f,\mathbf{m}} \frac{\inf\{\|D^n F\|_\infty : F \in L_\infty^{(n)}, F = f \text{ on } \mathbf{m}\}}{\max\{n!|[m_i, \ldots, m_{i+n}]f| : i \in \mathbb{Z}\}},$$

and studied the related optimization problem:

Minimize the norm of $F \in L_p^{(n)}$ *such that* $F = f$ *on* $\mathbf{m}$.

In an effort to generalize Favard's result to a multivariate setting, the author [8] proved a similar interpolation theorem in the simple case that the data function f was defined on a lattice in $\mathbb{R}^d$, that is, the image of $\mathbb{Z}^d$ under an invertible $d \times d$ matrix. In the special case that this matrix is diagonal, it was found that the d-fold tensor product of Favard's univariate operator produces an interpolant with many bounded derivatives: each (possibly mixed) partial derivative which is of degree at most n in each variable is bounded above by the corresponding tensor product divided differences of the data, times a constant $C(n, d)$.

More recently, however, efforts to find a similar result in the more general case that the data is given on a tensor product grid—that is, the Cartesian product of d increasing sequences of real numbers—led to a reexamination of the univariate problem and Favard's solution. It was discovered that Favard's interpolant does more than originally promised, since both its nth and $(n-1)$th derivatives are bounded by a constant times the corresponding divided differences (Theorem 3.1). It was also found that, however one forms an interpolant, one can bound at best two consecutive derivatives in terms of the corresponding divided differences (Theorem 3.7), so that, from this point of view, Favard's univariate result cannot be improved. This led directly to the identification of those derivatives which can be bounded (by corresponding divided differences times a grid-independent constant) during local interpolation on a tensor product grid (Theorems 4.2 and 4.3). These include the derivatives of total degree n only when $n = 1$, and even a weaker bound for these derivatives was found to be impossible if $n > 1$ (Theorem 4.5). The apparent lesson of all this would seem to be that a rectangular lattice is the only multidimensional data set on which the nth derivatives of a local interpolant can be made proportional to any or all of the nth tensor product divided differences of the data without a constant of proportionality that depends on the data set.

This work was prompted by the recent papers of Holtby [5, 6], who used the lattice results [8] to derive estimates for the solutions of nonlinear difference equations. Kreiss [7] had earlier used Favard's theorem to arrive at *à priori* error estimates for approximate solutions to ordinary differential equations, and Holby's results might lead to similar error bounds for partial differential equations.

We begin by establishing some notation in the next section.

2. Notation

The ith component of a point x in $\mathbb{R}^d$ is denoted $x(i)$.

For x and y in $\mathbb{R}^d$, let $[x, y]$ be the set of all u in $\mathbb{R}^d$ for which $x \le u \le y$ componentwise. The set of multiintegers is denoted $\mathbb{Z}^d$. In case x and y are in $\mathbb{Z}^d$, let $\{x, \ldots, y\}$ denote all the multiintegers in $[x, y]$. The multiindices are those multiintegers with nonnegative coordinates, collectively written $\mathbb{Z}^d_+$. If α is a multiindex, its length is $|\alpha| = \sum_i \alpha(i)$. We let $\mathbb{1}$ denote the vector $(1, 1, \ldots, 1)$ in $\mathbb{Z}^d$.

By $\mathbf{m}$ we shall mean a biinfinite, strictly increasing sequence of real numbers:

$$\mathbf{m} = \{\ldots, m_{-1}, m_0, m_1, m_2, \ldots\}$$

and by $\mathbf{M}$ we shall mean the Cartesian product of d such sequences

$$\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_d.$$

$\mathbf{M}$ is commonly called a tensor product grid. The elements of any one of these sequences is written with double subscripts, e.g.,

$$\mathbf{m}_1 = \{\ldots, m_{1,-1}, m_{1,0}, m_{1,1}, \ldots\}$$

and, for $z \in \mathbb{Z}^d$, the typical element of $\mathbf{M}$ is written

$$\mathbf{M}(z) := (m_{1,z(1)}, m_{2,z(2)}, \ldots, m_{d,z(d)}).$$

If f is a univariate function, then $[x_0, \ldots, x_n]f$ shall denote the nth divided difference of f at $\{x_0, \ldots, x_n\}$, with the usual meaning if some $x_i = x_j$. For $\mathbf{M}$ as above, $z \in \mathbb{Z}^d$, and $\alpha \in \mathbb{Z}_+^d$, define $\diamondsuit_{\mathbf{M},z}^{\alpha}$ to be the tensor product divided difference which acts of d-variate functions by applying $[m_{i,z(i)}, m_{i,z(i)+1}, \ldots, m_{i,z(i)+\alpha(i)}]$ in the ith variable.

3. The optimality of Favard's result

Theorem 3.1. *Let n be a natural number. For every strictly increasing, bi-infinite sequence of real numbers $\mathbf{m} := \{m_i : i \in \mathbb{Z}\}$, and every real-valued function f defined on $\mathbf{m}$, there exists an infinitely differentiable function $F_{\mathbf{m}}f$ defined on the interval $(\inf \mathbf{m}, \sup \mathbf{m})$ with the following properties.*

(1) $F_{\mathbf{m}}f$ agrees with f on $\mathbf{m}$.
(2) $F_{\mathbf{m}}f$ depends linearly on f.
(3) $F_{\mathbf{m}}f$ depends locally on f, in the sense that if $f(m_i)$ equals the Kronicker $\delta_{i,j}$ for some integer j, then the support of $F_{\mathbf{m}}f$ lies within $[m_{j-n}, m_{j+n}]$.
(4) There exists C independent of $\mathbf{m}$ and f such that, for $\nu = n$ or $\nu = n-1$, on each interval $[m_i, m_{i+1}]$,

$$|D^\nu F_{\mathbf{m}}f| \le C \max\Big\{ \big|[m_s, \ldots, m_{s+\nu}]f\big| : i - n < s \le i + n - \nu \Big\}.$$

In Favard's original theorem, the nth derivative of $F_{\mathbf{m}}f$ was only piecewise continuous, and item (4) was proven only for $\nu = n$. We obtain the stronger result with just a few minor changes to Favard's proof.

Proof: The interpolant $F_{\mathbf{m}}f$ is the same as one presented in an earlier paper [9] in which conclusions (1), (2), and (3) are proven.

In that proof, it is shown that, on the interval $[m_i, m_{i+1}]$, the function $F_{\mathbf{m}}f$—there written "Lf"—equals

$$F_{\mathbf{m}}f = P_j f + \sum_{s=j}^{h-1} \theta_h \psi_{s+1} \left([m_{s+1}, \cdots, m_{s+n}] - [m_s, \cdots, m_{s+n-1}]\right) f \tag{3.2}$$

$$= P_j f + \sum_{s=j}^{h-1} (m_{s+n} - m_s) \theta_h \psi_{s+1} [m_s, \cdots, m_{s+n}] f, \tag{3.3}$$

with the following explanations. First, j is an integer satisfying $j \le i < j + n$ and h is an integer satisfying $j \le h \le j + n$, so that the number of terms in the sum in (3.2) and (3.3) is between 0 and n. Second, $P_j f$ is the polynomial of degree less than n agreeing with f on $\{m_j, \ldots, m_{j+n-1}\}$. Third,

$$(m_{s+n} - m_s) \leq n(m_{i+1} - m_i) \tag{3.4}$$

for all s in the sum. Fourth, θ_h is a infinitely differentiable function. Fifth, ψ_{s+1} is the $(n-1)$th degree polynomial

$$\psi_{s+1} := (\cdot - m_{s+1}) \cdots (\cdot - m_{s+n-1}).$$

It is known [9] that there exist constants c_1 and c_2 independent of f and $\mathbf{m}$, so that, in the max-norm on $[m_i, m_{i+1}]$ and for all l between 0 and n inclusive, $\|D^l \theta_h\| = c_1 |m_{i+1} - m_i|^{-l}$ and $\|D^{n-l} \psi_{s+1}\| \leq c_2 |m_{i+1} - m_i|^{l-1}$. Consequently, for constants c_3 and c_4 independent of f and $\mathbf{m}$,

$$\|D^{n-1}(\theta_h \psi_{s+1})\| \leq c_3 \tag{3.5}$$

and

$$\|D^n(\theta_h \psi_{s+1})\| \leq c_4 |m_{i+1} - m_i|^{-1},$$

which, in light of (3.4), implies

$$\|(m_{s+n} - m_s) D^n(\theta_h \psi_{s+1})\| \leq n c_4. \tag{3.6}$$

Differentiating (3.2) $n-1$ times and combining the result with (3.5) proves (3) in case $\nu = n-1$. Differentiating (3.3) n times and applying (3.6) gives (3) in case $\nu = n$, completing the proof of Theorem 3.1. ■

The next theorem concerns univariate interpolation operators in general, and the impossibility of bounding more than two (consecutive) derivatives by corresponding divided differences.

Theorem 3.7. *Let n be a natural number and F be an operator that maps every biinfinte, strictly increasing sequence of real numbers $\mathbf{m}$ and function f defined on $\mathbf{m}$ to a smooth function*

$$F : (\mathbf{m}, f) \mapsto F_{\mathbf{m}} f \in C^n(\inf \mathbf{m}, \sup \mathbf{m})$$

agreeing with f on $\mathbf{m}$, i.e. $F_{\mathbf{m}} f(m_i) = f(m_i)$ for all i. Suppose that there exists a constant C_n such that, for all such $\mathbf{m}$ and f, and for every $n+1$ not-necessarily-distinct points $u_0, \ldots, u_n$ in $\mathbb{R}$,

$$|[u_0, \ldots, u_n] F_{\mathbf{m}} f| \leq C_n \sup_i |[m_i, \ldots, m_{i+n}] f|. \tag{3.8}$$

Then there exists $f \in C(\mathbb{R})$ such that, for every k in $\{0, \ldots, n-2\}$, and for every constant C_k, there exists $\mathbf{m}$ and $u_0, \ldots, u_k$ such that

$$|[u_0, \ldots, u_k] F_{\mathbf{m}} f| > C_k \sup_i |[m_i, \ldots, m_{i+k}] f|. \tag{3.9}$$

That is, if the nth divided differences of $F_{\mathbf{m}}f$ are bounded by a constant times their necessary size, with the constant independent of f and $\mathbf{m}$, then the same cannot be true of the kth divided differences for any k less than $n-1$. (Note that to bound $F_{\mathbf{m}}f$'s divided differences of any particular degree is equivalent to bounding its corresponding derivative.)

In light of Theorems 3.1 and 3.7, Favard's result is optimal, in the sense that, among all ways of producing interpolants whose derivatives are absolutely bounded by the data's corresponding divided differences times constants independent of $\mathbf{m}$ and f, including nonlinear or nonlocal schemes, Favard's method bounds as many derivatives as possible.

Proof of Theorem 3.7: The theorem is vacuously true if $n = 1$, so assume that $n \geq 2$. Define the function f as

$$f(x) = \begin{cases} x^{n-1} & \text{if } 0 \leq x < 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For any ϵ in $(0,1)$, let $\mathbf{m}$ be a sequence

$$\ldots, m_{-2} = -2, m_{-1} = -1, m_0 = 0 < m_1 < m_2 < \cdots < m_{n-1} = \epsilon, m_n = 1, m_{n+1} = 2, \ldots$$

The proof of Theorem 3.7 depends on the next three lemmas.

Lemma 3.10. *If* $0 \leq k \leq n-2$, *then* $\lim_{\epsilon \to 0} \sup_i |[m_i, \ldots, m_{i+k}]f| = 0$.

Proof of Lemma 3.10: For any ϵ, only finitely many of the kth divided differences $[m_i, \ldots, m_{i+k}]f$ are nonzero, and as ϵ goes to zero, each approaches one of the form

$$[\ldots, -2, -1, \underbrace{0, 0, \ldots, 0}_{\leq k+1}, 1, 2, \ldots]f,$$

a linear combination of f's values at the integers, and the values of $f, f', \ldots, f^{(k)}$ at zero, all of which are zero. ■

Lemma 3.11. *For all* ϵ *sufficiently close to zero,* $\sup_i |[m_i, \ldots, m_{i+n}]f| \leq 2$.

Proof of Lemma 3.11: For any ϵ, only finitely many of the nth divided differences $[m_i, \ldots, m_{i+n}]f$ are nonzero, and as ϵ goes to zero, each approaches one of the form

$$[\ldots, -2, -1, \underbrace{0, 0, \ldots, 0}_{\leq n}, 1, 2, \ldots]f,$$

a linear combination of f's values at the integers, the values of $f, f', \ldots, f^{(n-2)}$ at zero, and the righthand limit of $f^{(n-1)}$ at zero. Of these, only $f^{(n-1)}(0^+)$ is nonzero. Hence, the only nonzero limits are

$$[-1, \underbrace{0, 0, \ldots, 0}_{n}]f = 1, \qquad \text{and} \qquad [\underbrace{0, 0, \ldots, 0}_{n}, 1]f = -1,$$

proving Lemma 3.11. ■

The final lemma before the proof of Theorem 3.7 involves the limit behavior of the divided differences of $F_{\mathbf{m}}f$.

Lemma 3.12. *If $m_i, \ldots, m_{i+l-1}$ are l points within ϵ of zero, and if x is a number greater than ϵ, then*

$$[m_i, \ldots, m_{i+l-1}, x]F_{\mathbf{m}}f = \begin{cases} x^{-l}F_{\mathbf{m}}f(x) + \mathrm{o}(1) & \text{if } l < n, \text{ and} \\ x^{-n}F_{\mathbf{m}}f(x) - x^{-1} + \mathrm{o}(1) & \text{if } l = n. \end{cases}$$

where o(1) *denotes a quantity depending on x which approaches zero as ϵ goes to zero.*

Proof of Lemma 3.12: First note that, for any x as in the hypotheses, $F_{\mathbf{m}}f(x)$ remains bounded as $\epsilon \to 0$. Indeed, by Lemma 3.11 and Equation (3.8) , for ϵ sufficiently close to zero,

$$\begin{aligned} |F_{\mathbf{m}}f(x)| &= x(x+1)\cdots(x+n-1)|[-n+1, \ldots, -1, 0, x]F_{\mathbf{m}}f| \\ &\le x(x+1)\cdots(x+n-1)2C_n. \end{aligned}$$

To prove Lemma 3.12, induct on l, the case $l = 0$ being trivial. For the inductive step, use the recurrence

$$[m_i, \ldots, m_{i+l-1}, x]F_{\mathbf{m}}f = \frac{[m_{i+1}, \ldots, m_{i+l-1}, x]F_{\mathbf{m}}f - [m_i, \ldots, m_{i+l-1}]F_{\mathbf{m}}f}{x - m_i}.$$

Since $F_{\mathbf{m}}f = f$ on $\mathbf{m}$, the inductive hypothesis and Rolle's Theorem imply that this equals

$$\big(x^{1-l}F_{\mathbf{m}}f(x) + \mathrm{o}(1) - f^{(l-1)}(\xi)/(l-1)!\big)/(x - m_i)$$

for some ξ between 0 and ϵ. As ϵ goes to zero, the derivative $f^{(l-1)}(\xi)$ converges to $(l-1)!$ if $l = n$ and to zero otherwise, while the boundedness of $F_{\mathbf{m}}f(x)$ implies

$$\frac{x^{1-l}F_{\mathbf{m}}f(x)}{x - m_i} = x^{-l}F_{\mathbf{m}}f(x) + \mathrm{o}(1),$$

completing the proof of Lemma 3.12. ■

To finish the proof of Theorem 3.7, recall hypothesis (3.8) and choose a positive $\epsilon_0 < 1/(2C_n)$ such that $\epsilon < \epsilon_0$ will ensure that $\sup_i |[m_i, \ldots, m_{i+n}]f| \le 2$, as guaranteed by Lemma 3.11. By hypothesis (3.8),

$$[m_0, \ldots, m_{n-1}, x]F_{\mathbf{m}}f \ge -C_n.$$

for any real number x. Fix an x in $\big(\epsilon_0, 1/(2C_n)\big)$. By Lemma 3.12, there is a positive $\epsilon_1 < \epsilon_0$ such that

$$\big|[m_0, \ldots, m_{n-1}, x]F_{\mathbf{m}}f - (x^{-n}F_{\mathbf{m}}f(x) - x^{-1})\big| < C_n$$

for all $\epsilon < \epsilon_1$, and therefore

$$\frac{F_{\mathbf{m}}f(x)}{x^n} - \frac{1}{x} \ge -2C_n. \tag{3.13}$$

The proof is by contradiction. Assume that the conclusion of Theorem 3.7 is false, that is, for some $k \le n-2$, the kth divided differences of $F_{\mathbf{m}}f$ are bounded by the kth divided differences of f on $\mathbf{m}$ times a constant independent of $\mathbf{m}$. Lemma 3.10 then implies

$$\lim_{\epsilon\to 0}[m_0,\ldots,m_{k-1},x]F_{\mathbf{m}}f = 0,$$

and Lemma 3.12 therefore implies

$$\lim_{\epsilon\to 0}\frac{F_{\mathbf{m}}f(x)}{x^k} = \frac{1}{x^k}\lim_{\epsilon\to 0}F_{\mathbf{m}}f(x) = 0.$$

Applying this limit to (3.13) contradicts $x < 1/(2C_n)$ and proves Theorem 3.7. ■

4. Interpolation on lattices and tensor product grids

In an effort to find a Favard-like interpolant for functions of several variables, the author [8,(5.2)] proved the following theorem for the special case that the data set $\mathbf{M}$ is a rectangular lattice, that is, a tensor product grid in which every coordinate sequence $\mathbf{m}_i$ is arithmetic and each $m_{i,0} = 0$. Recall that $\mathbb{1} = (1,1,\ldots,1)$.

Theorem 4.1. *For every nonnegative integer n and positive integer d, there exists a mapping F from the set of all rectangular lattices $\mathbf{M}$ and functions f defined on $\mathbf{M}$,*

$$F : (\mathbf{M}, f) \mapsto F_{\mathbf{M}}f \in C^\infty(\mathbb{R}^d)$$

with the following properties.

(1) $F_{\mathbf{M}}f$ agrees with f on $\mathbf{M}$.
(2) $F_{\mathbf{M}}f$ depends linearly on f.
(3) $F_{\mathbf{M}}f$ depends locally on f in the sense that, if $f\big(\mathbf{M}(u)\big) = \delta_{u,v}$ and all u and some v in $\mathbb{Z}^d$, then the support of $F_{\mathbf{M}}f$ is contained within the box $[\mathbf{M}(v-n\mathbb{1}), \mathbf{M}(v+\mathbb{1})]$.
(4) There exists a constant C such that for all $\mathbf{M}$ and f, and for all $v \in \mathbb{Z}^d$ and multi-indices $\alpha \le n\mathbb{1}$,

$$\max\{|D^\alpha F_{\mathbf{M}}f(x)| : x \in [\mathbf{M}(v), \mathbf{M}(v+\mathbb{1})]\}$$
$$\le C\max\{|\Diamond^\alpha_{\mathbf{M},u}f| : \operatorname{supp}\Diamond^\alpha_{\mathbf{M},u} \subset [\mathbf{M}(v), \mathbf{M}(v+n\mathbb{1})] \cap \mathbf{M}\}$$

This result can be generalized [8, 6.4] to the case when $\mathbf{M}$ is a nonrectangular lattice in $\mathbb{R}^d$.

The operator F in Theorem 4.1 is simply the tensor product of univariate Favard operators. That so many derivatives are each bounded in terms of the corresponding divided differences is a special consequence of $\mathbf{M}$'s being a lattice. More generally, when $\mathbf{M}$ is a tensor product grid , the maximal set of derivatives which can be bounded during local interpolation—linear or not—is described in the next two theorems.

Theorem 4.2. *Let n and d be positive integers. Suppose F is an interpolation operator mapping data f defined on a tensor product grids $\mathbf{M}$ to functions $F_{\mathbf{M}}f \in C^n(\mathbb{R}^d)$ satisfying the following three conditions.*

(1) $F_{\mathbf{M}}f = f$ on $\mathbf{M}$.

(2) $F_{\mathbf{M}}f$ depends locally on f, in the sense that there exists a natural number N such that, for all $\mathbf{M}$ and f, if

$$(\operatorname{supp} f)_i \subset \{m_{i,j}, \dots, m_{i,k}\}$$

then

$$(\operatorname{supp} F_{\mathbf{M}}f)_i \subset [m_{i,j-N}, m_{i,k+N}].$$

(Here, $(X)_i$ is the orthogonal projection of $X \subset \mathbb{R}^d$ onto the ith coordinate axis.)

(3) For some constant C independent of $\mathbf{M}$ and f,

$$\|D^\alpha F_{\mathbf{M}}f\|_{L_\infty(\mathbb{R}^d)} \le C\|\diamondsuit^\alpha_{\mathbf{M}} f\|_{l_\infty(\mathbb{Z}^d)}$$

and

$$\|D^\beta F_{\mathbf{M}}f\|_{L_\infty(\mathbb{R}^d)} \le C\|\diamondsuit^\beta_{\mathbf{M}} f\|_{l_\infty(\mathbb{Z}^d)}.$$

Then $\max_i |\alpha(i) - \beta(i)| \le 1$.

For instance, it is impossible for a interpolant $F_{\mathbf{M}}f$ to data f on a tensor product grid $\mathbf{M}$ in $\mathbb{R}^2$ to satisfy both

$$\|D^{(2,0)} F_{\mathbf{M}}f\|_{L_\infty(\mathbb{R}^2)} \le C\|\diamondsuit^{(2,0)}_{\mathbf{M}} f\|_{l_\infty(\mathbb{Z}^2)}$$

and

$$\|D^{(0,2)} F_{\mathbf{M}}f\|_{L_\infty(\mathbb{R}^2)} \le C\|\diamondsuit^{(0,2)}_{\mathbf{M}} f\|_{l_\infty(\mathbb{Z}^2)}$$

unless the constant C depends on $\mathbf{M}$ or f, or the interpolation scheme is not local. (The localness assumed in Theorem 4.2 is similar to that of Favard's theorem, in that support is bounded in terms of an absolute number of steps in the data sequences.) Theorem 4.3 will show that the maximal set of bounded derivatives allowed by Theorem 4.2, $\{D^\beta : \beta \in \{\alpha - \mathbb{1}, \dots, \alpha\}\}$, is attainable for arbitrary α.

Proof of Theorem 4.2: The proof is by contradiction. Suppose, without loss of generality, that $k := \alpha(1) < \beta(1) - 1 =: n - 1$, and define

$$f : \mathbb{R}^d \to \mathbb{R} : x \mapsto \begin{cases} x(1)^{n-1} & \text{if } 0 \le x(1) < 1 \text{ and } x(2) = \cdots = x(d) = 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$, define $\mathbf{M}$ by choosing $\mathbf{m}_1$ to be the sequence $\mathbf{m}$ from the proof of Theorem 3.7, and $\mathbf{m}_2 = \cdots = \mathbf{m}_d = \mathbb{Z}$. Then, according to Lemmas 3.10 and 3.11, as ϵ goes to zero, the divided differences $\diamondsuit^\alpha_{\mathbf{M},z} f$ (defined in the last sentence of Section 2) converge to zero uniformly in z, while the differences $\diamondsuit^\beta_{\mathbf{M},z} f$ remain bounded. Let $\epsilon < x < 1$, and consider the αth tensor product divided difference

$$[m_{1,1}, \dots, m_{1,k}, x] \otimes [0, \underbrace{N, \dots, N}_{\alpha(2)}] \otimes \cdots \otimes [0, \underbrace{N, \dots, N}_{\alpha(d)}] F_{\mathbf{M}}f.$$

Since $(\operatorname{supp} F_{\mathbf{M}} f)_i \subset [-N,N]$ for each $i \in \{2,\ldots,d\}$, this divided difference equals

$$\frac{1}{N^{\alpha(2)+\cdots+\alpha(d)}}[m_{1,1},\ldots,m_{1,k},x]F_{\mathbf{M}}f(\cdot,\underbrace{0,0,\cdots,0}_{d-1}).$$

As in the proof of Theorem 3.7, Lemma 3.12 now implies

$$\lim_{\epsilon\to 0} F_{\mathbf{M}}f(x,\underbrace{0,0,\cdots,0}_{d-1}) = 0$$

and yet, for some at least one fixed x, $F_{\mathbf{M}}f(x,0,0,\cdots,0)$ is bounded away from zero as $\epsilon \to 0$, a contradiction, proving Theorem 4.2. ■

Theorem 4.3. *For every natural number d, and multiindex $\alpha \geq \mathbb{1}$, there exists a positive constant C and a mapping F from the set of all tensor product grids $\mathbf{M}$ and functions f defined on $\mathbf{M}$*

$$F : (\mathbf{M}, f) \mapsto F_{\mathbf{M}}f \in C^\infty(\mathbb{R}^d)$$

satisfying the following conditions.

(1) $F_{\mathbf{M}}f = f$ on $\mathbf{M}$,

(2) $F_{\mathbf{M}}f$ depends linearly and locally on f, in the sense that, for all $\mathbf{M}$ and f and any v in $\mathbb{Z}^d$, if $f\big(M(u)\big)$ equals the Kronicker $\delta_{u,v}$ for all u in $\mathbb{Z}^d$, then the support of $F_{\mathbf{M}}f$ is contained in the box $[M(v-\alpha), M(v+\alpha)]$.

(3) For all $\mathbf{M}$ and f and all $v \in \mathbb{Z}^d$ and any multiindex β in $\{\alpha - \mathbb{1}, \ldots, \alpha\}$

$$\max\big\{|D^\beta F_{\mathbf{M}}f(x)| : x \in [\mathbf{M}(v), \mathbf{M}(v+\mathbb{1})]\big\}$$
$$\leq C \max\big\{|\Diamond^\beta_{\mathbf{M},u} f| : v-\alpha < u \leq v+\alpha-\beta\big\}.$$

Proof: Take $F_{\mathbf{M}}$ to be the tensor product $F_1 \otimes \cdots \otimes F_d$ of d copies of Favard's operator F from Theorem 3.1 using $n = \alpha(k)$ in the construction of F_k. Conclusions (1) and (2) follow from Theorem 3.1. To prove (3), assume without loss of generality that $v = 0$.

As is the proof of Theorem 3.1, on the interval $[m_{k,o}, m_{k,1}]$, the function produced by F_k is the same as that produced by

$$\begin{aligned} &P_j + \sum_{s=j}^{h-1}(m_{k,s+\alpha(k)} - m_{k,s})\theta_h\psi_{s+1}[m_{k,s},\ldots,m_{k,s+\alpha(k)}] \\ &= P_j + \sum_{s=j}^{h-1}\theta_h\psi_{s+1}([m_{k,s+1},\ldots,m_{k,s+\alpha(k)}] - [m_{k,s},\ldots,m_{k,s+\alpha(k)-1}]) \end{aligned} \tag{4.4}$$

where j is in $\{1-\alpha(k),\ldots,0\}$,

$$P_j = (\cdot)^{\alpha(k)-1}[m_{k,j}, m_{k,j+1}, \ldots, m_{k,j+\alpha(k)-1}] + \text{lower order terms},$$

h is in $\{j, \ldots, j+\alpha(k)\}$, and θ_h and ψ_{s+1} are smooth functions of the kth variable. Consequently, the restriction of $(F_1 \otimes \cdots \otimes F_d)f$ to the box $[\mathbf{M}(0), \mathbf{M}(\mathbb{1})]$ can be written

$$\sum_{\substack{u \in \mathbb{Z}^d \\ \gamma \leq \beta}} \Lambda_{\mathbf{M},u,\gamma} \diamondsuit^{\gamma}_{\mathbf{M},u} f,$$

a finite sum of smooth functions times tensor product divided differences of f. The divided differences of degree γ less than β are present due to the lower order differences and polynomials in P_j.

If $\gamma(k) < \beta(k)$ for some k, then, in the kth variable, $\Lambda_{\mathbf{M},u,\gamma}$ is a polynomial of degree $\gamma(k)$, so $D^{\beta}\Lambda_{\mathbf{M},u,\gamma} = 0$. The remaining functions $\Lambda_{\mathbf{M},u,\beta}$ are, in the kth variable, a polynomial of degree $\alpha(k)$ plus, in case the sum in (4.4) is nonempty, functions whose $\beta(k)$th derivatives are bounded on $[m_{k,0}, m_{k,1}]$ by equations (3.5) and (3.6). This proves (3), except for the detail $v - \alpha < u \leq v + \alpha - \beta$, which is obtained from the corresponding inequalities in (3) of Theorem 3.1. ■

One consequence of Theorem 4.2 is that, if n and d both exceed 1, then no local interpolation scheme for tensor product grids can bound each derivative of total degree n in terms of its corresponding divided difference without a constant that depends on the grid $\mathbf{M}$. This leaves open the natural question of whether a weaker bound might still be possible; in his application [5, 6] of Theorem 4.1, Holtby required only that the *largest* nth derivative be bounded by the *largest* nth divided difference (as in (3) below) times a constant independent of the mesh $\mathbf{M}$. The final theorem of this section, however, shows that such a bound, though far weaker than the bound in Theorem 4.1, is still not possible for a local interpolant on tensor product grids.

Theorem 4.5. *Let n and d be integers greater than 1. Suppose F is an operator mapping functions f defined on tensor product grids $\mathbf{M}$ in $\mathbb{R}^d$ to functions $F_{\mathbf{M}}f \in C^n(\mathbb{R}^d)$ satisfying the following three conditions.*

(1) $F_{\mathbf{M}}f = f$ on $\mathbf{M}$.

(2) $F_{\mathbf{M}}f$ depends locally on f, in the sense that there exists a positive integer N independent of $\mathbf{M}$ and f such that the restriction of $F_{\mathbf{M}}f$ to any box $[M(v), M(v+\mathbb{1})]$ depends only on f's values at the nearby grid points $\{M(u) : u \in \{v - N\mathbb{1}, \ldots, v + N\mathbb{1}\}\}$.

(3) For some constant C independent of f,

$$\max_{|\alpha|=n} \|D^{\alpha}F_{\mathbf{M}}f\|_{L_\infty(\mathbb{R}^d)} \leq C \max_{|\alpha|=n} \|\diamondsuit^{\alpha}_{\mathbf{M}}f\|_{l_\infty(\mathbb{Z}^d)}\,.$$

Then C cannot be independent of $\mathbf{M}$.

Proof: The proof is by contradiction. Assume (1), (2), and (3) are true with C independent of $\mathbf{M}$. If such an operator exists on $\mathbb{R}^d$ for some $d > 2$, then one must also exist on $\mathbb{R}^2$, since any $\mathbf{M}$ and f in $\mathbb{R}^2$ can be extened to $\mathbb{R}^d$ by making f constant in $d-2$ directions, and the only nonzero differences of such an f would be obtained from the original data in $\mathbb{R}^2$. Therefore, assume $d = 2$.

Let $0 < \epsilon < 1$, and define $\mathbf{M}$ to be the tensor product grid in $\mathbb{R}^2$ whose coordinate sequences are

$$\mathbf{m}_1 = \{\cdots, -3\epsilon, -2\epsilon, -\epsilon, 0, \epsilon, 2\epsilon, 3\epsilon, \cdots\}$$

and

$$\mathbf{m}_2 = \{\cdots, -3, -2, -1, -\epsilon, 0, 1, 2, 3, \cdots\}.$$

Define

$$f(x,y) = \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are} \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Condition (3) implies that, even if $F_{\mathbf{M}}$ is nonlinear, $F_{\mathbf{M}}0 = 0$, since $F_{\mathbf{M}}$ reproduces all polynomials of degree less than n on $\mathbf{M}$. The localness of $F_{\mathbf{M}}$ therefore implies that $F_{\mathbf{M}}f(x,y)$ is identically zero on the half-plane $x < -(N+1)\epsilon$. Another consequence of localness is that, for x sufficiently greater than zero, e.g. $x = N\epsilon$, the interpolant $F_{\mathbf{M}}f(x,y)$ is the same as $F_{\mathbf{M}}g(x,y)$, where

$$g(x,y) = \begin{cases} 1 & \text{if } y \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since the nth total-degree divided differences of f are $\mathrm{O}(\epsilon^{-n})$ and C is assumed independent of ϵ, the nth total-degree derivatives of $F_{\mathbf{M}}f$ are also $\mathrm{O}(\epsilon^{-n})$. Additionally, since the nth divided differences of g are only $\mathrm{O}(\epsilon^{-1})$, the nth derivative of $F_{\mathbf{M}}f$ in the y-direction along the line $x = N\epsilon$ is at most $\mathrm{O}(\epsilon^{-1})$.

Consequently, for some positive constant K and any y,

$$\begin{aligned} \frac{K}{\epsilon^n} &\geq [\underbrace{-(N+2)\epsilon, -(N+2)\epsilon, \ldots, -(N+2)\epsilon}_{n}, N\epsilon]F_{\mathbf{M}}f(\cdot, y) \\ &= \frac{F_{\mathbf{M}}f(N\epsilon, y)}{(2N+2)^n\epsilon^n} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \frac{-K}{\epsilon} &\leq [-n+2, \ldots, -1, 0, -\epsilon, y]F_{\mathbf{M}}f(N\epsilon, \cdot) \\ &= \frac{F_{\mathbf{M}}f(N\epsilon, y)}{(y+\epsilon)y(y+1)(y+2)\cdots(y+n-2)} - \frac{1}{\epsilon y(n-2)!}. \end{aligned}$$

If $y > 0$, then

$$\epsilon F_{\mathbf{M}}f(N\epsilon, y) \geq \left(\frac{1}{(n-2)!} - Ky\right)(y+\epsilon)(y+1)\cdots(y+n-2).$$

Choose a positive y so that the limit L of the right side is positive. For all ϵ sufficiently close to zero, equation (4.6) implies

$$\frac{K}{\epsilon^n} \geq \frac{L/2}{(2N+2)^n\epsilon^{n+1}}.$$

Letting ϵ go to zero arrives at a contradiction and completes the proof of Theorem 4.5. ■

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